THE ELEMENTS OF PHYSICS

VOL. I

MECHANICS AND HEAT

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THE

ELEMENTS OF PHYSICS

A COLLEGE TEXT-BOOK

BY

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AND

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IN THREE VOLUMES

VOL. I

MECHANICS AND HEAT

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PREFACE.

The beginner in Physics, if he is to be well taught, must study a text-book which contains concise statements of physical laws and a systematic development of principles. He must attend lectures or demonstrations in which the phenomena are shown, and finally he must enter the laboratory and make physical measurements.

The present book deals only with the first part of this triple course of instruction. It differs in certain respects from many books which have been written for beginners. Descriptions of the numerous phenomena with which it is necessary for the student to become acquainted have been largely omitted; the assumption being that the study of the text will always be accompanied by, and indeed be supplementary to, a course of carefully arranged and fully illustrated lectures. A knowledge of the elements of the calculus, the natural language of physics, has been taken for granted, but the degree of mathematical experience of the undergraduate reader, necessarily limited, has been kept in view and the various proofs and the demonstrations have been given the simplest possible form.

The concepts of directed and of distributed quantity, which are no less important to the student of physics than are the methods of the calculus, are briefly treated in Chapter II. of Volume I. and in Chapter I. of Volume II., respectively, and are used upon occasion throughout the text.

The authors have thought it best to face squarely the difficulties of the subject, and they may perhaps be permitted to say with Tait:[*] "He who expects to find this work, elementary as it is, everywhere easy reading, will be deservedly disappointed. No branch of science is free from real and great difficulties even in its elements. Any one who thinks otherwise has either not read at all or has confined his reading to *pseudo*-science."

What follows the word *Law*, as used in this text, is in every case an independent statement of experimental fact. Many physical laws are the direct result of experiment, others, such as Newton's Laws of Motion, the Laws of Thermodynamics, etc., are in their accepted form more or less indirectly derived from experiment, and are so comprehensive as to defy complete experimental verification. The principle that the only proper discussion of a law is that which is necessary to make its meaning clear has been closely followed. The theoretical considerations which follow the statement of a number of laws referring to the same thing, for example the discussion of the Kinetic theory, which follows the statements of the various laws of gases, have in every case the object of forming concrete pictures of the abstractions contemplated in the laws. Helmholtz[†] has long since pointed out the advantages of this method of treatment.

The work is issued in three small volumes, the first of which treats of Mechanics and Heat, the second of Electricity and Magnetism, and the last of Sound and Light. As a matter of convenience, articles, equations, and figures are numbered consecutively throughout the three volumes.

ITHACA, NEW YORK, Dec. 3, 1895.

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[*] Tait, Heat, p. viii of the preface.

[†] "Es hat grosse Vortheile für das sichere Verständniss solcher Abstractionen, wenn man sich möglichst concrete Bilder davon zu machen sucht, selbst wenn diese manche Voraussetzung hineinbringen die für das Wesen der Sache nicht gerade nothwendig ist." — von Helmholtz, Handbuch der physiologischen Optik.

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THE ELEMENTS OF PHYSICS.

CHAPTER I.

MEASUREMENT OF LENGTH, TIME, AND MASS.

1. Length: the meter and the yard. — The *Meter* is defined as the distance, at the temperature of melting ice, between the ends of a platinum bar preserved in the Archives of France. The centimeter, which in physics is adopted as a fundamental unit, is one hundredth of a meter. This bar was made by Borda to be the ten-millionth part of the distance from the equator to the pole of the earth. The measurements upon which it was based are now known to be distinctly incorrect, which fact, together with the fact that copies of the standard meter can be made with greater accuracy and convenience by direct comparison with Borda's bar than by comparison with the earth's quadrant has resulted in the adoption of the above definition of the meter.

The *Yard* is defined by act of Parliament as the distance at 62° F. between the transverse lines in two gold plugs in a bronze bar deposited in the office of the Exchequer in London. The yard is equal to 0.914383 meter.

2. The measurement of length: scale and vernier. — Lengths are commonly measured by the use of divided scales. In this case the length to be measured is determined by counting the number of whole divisions and estimating the fractions remaining over. The latter process is sometimes performed by means of the *Vernier*, which is a device for the accurate estimation of fractions of divisions in the use of straight scales and divided circles. Let S in Fig. 1 be a scale divided to millimeters. An auxiliary scale, V, $(n-1)$ millimeters long, is divided into n equal parts. The diagram (Fig. 1) is constructed for $n=10$.

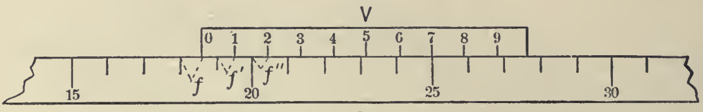

Fig. 1.

Let f, equal to $\frac{a}{n}$ mm., be the fraction to be estimated; f' is $\frac{1}{n}$ mm. shorter, f'' is $\frac{2}{n}$ mm. shorter, and so on, so that the ath mark on V is coincident with a mark on the scale. Thus a is determined. The auxiliary scale, V, is called a vernier. The n divisions on the vernier may be made each $\frac{n+1}{n}$ mm., in which case a is determined by counting from the other end of the vernier.

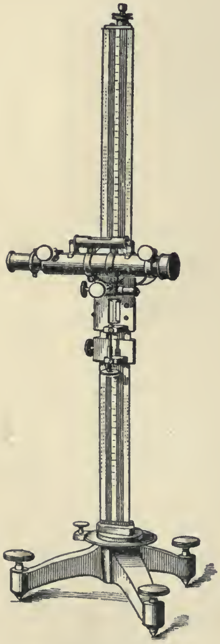

Fig. 2.

3. The cathetometer (Fig. 2) is an instrument for measuring differences in level. It consists of a vertical scale upon which is a sliding piece, with clamp, carrying a telescope accurately leveled by means of a sensitive spirit level, and a vernier which plays over the vertical scale. The telescope is sighted at one of the two points of which the difference in level is required and the reading of the vernier taken. The telescope is then sighted at the other point and the vernier reading again taken. The difference of these readings is the required difference in level.

4. For the more precise determination of lengths a variety of instruments is employed which depend for their action upon the properties of the screw.

The dividing engine is such an instrument. It consists essentially of an accurate horizontal screw,* having a divided circular head for estimating fractions of a turn. This screw engages a nut fastened to a carriage, which carries a reading microscope and a graving tool, and slides upon a heavy metal bed. The standard of length is placed upon this platform parallel to the screw, and the screw is turned until the microscope sights exactly at one end of it. The screw is then turned, *a* turns, until the microscope sights at the other end of the standard. The bar the length of which is to be determined is put in place of the standard, the screw is turned until the microscope sights at one end of the bar and then, *b* turns, until the microscope sights at the other end of the bar. The length of the bar is then known to be to the standard as $\frac{b}{a}$ is to 1. In the manufacture of scales a blank bar is put upon the platform, the screw is turned until the graving tool is conveniently near one end of the bar and a mark is made; the screw is turned $\frac{a}{n}$ turns, and another mark is made, and so on, thus dividing the bar into *n*ths of the standard length. In use, the dividing engine must be carefully maintained at a constant temperature. Where the lengths to be determined are small, the screw thus employed is called a micrometer screw.

The *micrometer screw* is accurately cut to some definite pitch. It has a divided circular head and turns in a fixed, close-fitting nut. The following are two well-known forms.

(*a*) The *micrometer caliper* is a micrometer screw mounted as shown in Fig. 3.

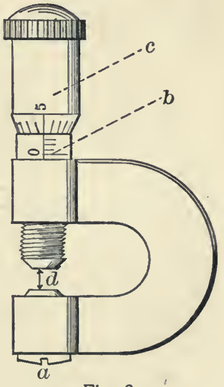

Fig. 3.

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* See Encyclopedia Britannica, 9th ed., article *Screw*.

The instrument is so adjusted that the distance *d* is indicated directly by the screw, the whole turns upon a fixed scale *b*, and the fractions of a turn upon the circular head *c*.

(*b*) The *spherometer* (Fig. 4) is a micrometer screw the nut of which is carried on a rigid metal tripod so that the screw may be perpendicular to a plane upon which the tripod stands. The screw carries a conical tip the apex of which is equidistant from the feet of the tripod, and these are equidistant from each other. The instrument is used for determining the radii of curvature of spherical surfaces, for which purpose a reading is taken when the screw is just in contact with a plane surface upon which the instrument stands, and again when it is placed upon the given spherical surface. From these readings, the pitch of the screw and the distance between the feet of the tripod being known, the radius of the surface is easily calculated.

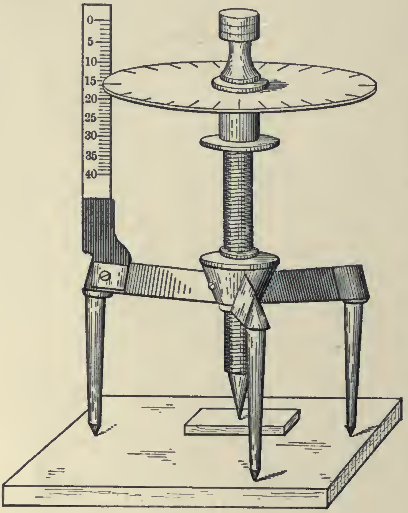

Fig. 4.

5. Angle. — Angles are ordinarily specified in degrees, the degree being the unit upon which trigonometrical tables are based. It is often more convenient, however, to express angle in terms of $\frac{\text{arc}}{\text{radius}}$. The unit angle in this system is called the *Radian*, and is the angle of which the arc is equal to the radius.

Measurement of angles. — In many instruments angles are measured by means of a divided circle. This is placed with its center at the apex of the angle, which is measured by counting the divisions on the circle between the lines which determine the angle. These lines are established by using a pair of sights, or a telescope, fixed to an arm, called an alidade. The latter turns on a pivot at the center of the circle, and carries a vernier playing over the divided circle. Frequently, two verniers situated 180° apart are used to eliminate errors due to inaccurate centering of the pivot.

Indirect measurement of length and angle. — Consider a plane triangle of which A, B, C are the angles, and a, b, c are the sides.

We have $$A + B + C = 180°,$$

$$a \sin B = b \sin A, \qquad (1)$$

$$a^2 = b^2 + c^2 - 2\,bc \cos A.$$

These are three independent equations always satisfied by these six quantities. If three of these quantities have been measured, one of which, at least, is a side, the others may be calculated.

The only case of the above indirect method for measuring angle which is used, to any great extent, in the physical laboratory, is known as *Poggendorff's method.* It is as follows :

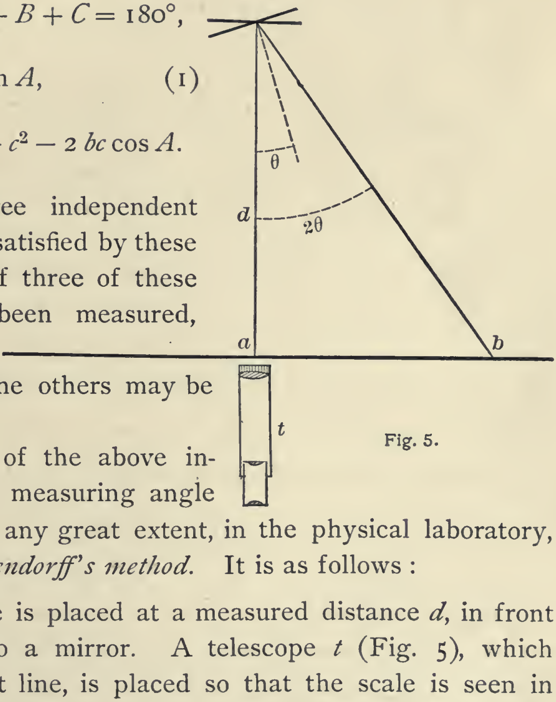

Fig. 5.

A straight scale is placed at a measured distance d, in front of and parallel to a mirror. A telescope t (Fig. 5), which establishes a sight line, is placed so that the scale is seen in the mirror, the sight line being perpendicular to the scale. The reading a of this sight line on the scale is taken; the mirror then turns through the angle θ to be measured, and the scale reading, b, is again taken. The sight line is evidently deflected through 2θ, so that $\tan 2\theta = \frac{a-b}{d}$, from which θ may be calculated.

6. Area. — The unit of area is defined as the area of a square of which the side is unit length. Area is determined, fundamentally, by calculation from measured linear dimensions.

The *planimeter* is an instrument for measuring irregular plane areas.

Consider a line AB (Fig. 6), of length l, moving in any manner in the plane of the paper. The motion may at each instant be considered as compounded of a motion of translation and a motion of rotation, with angular velocity $\frac{d\phi}{dt}$, about an arbitrary point p distant d from the center of the line. See Art. 82.

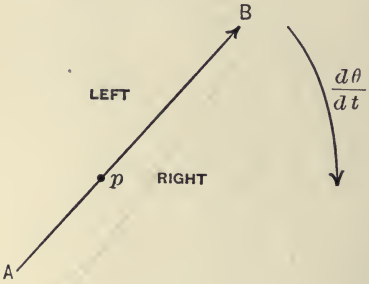

Fig. 6.

The area swept by this line is considered positive when the line sweeps over it from left to right to an observer looking from A to B. Let v be the resolved part, perpendicular to the line, of its velocity of translation. The line sweeps over area at the rate lv because of its motion of translation, and at a rate $ld\frac{d\phi}{dt}$ because of its motion of rotation, so that the total rate at which the line sweeps area at each instant is

$$\frac{dA}{dt} = lv + dl\frac{d\phi}{dt}. \tag{2}$$

Let a wheel, radius r, mounted at p, with its axis parallel to AB, be allowed to roll on the paper as the line moves, and let $\frac{d\psi}{dt}$ be the angular velocity of rolling of the wheel.

Then $v = r\frac{d\psi}{dt}$, and equation (2) becomes

$$\frac{dA}{dt} = lr\frac{d\psi}{dt} + ld\frac{d\phi}{dt},$$

or

$$A = lr\psi + dl\phi; \tag{3}$$

A, ψ, and ϕ being reckoned from the initial position of the line for which $A=0$, $\psi=0$, and $\phi=0$. If the line comes back to its initial position, or parallel thereto, so that $\phi=0$, then equation (3) becomes $A=lr\psi$, and the circumference of the wheel may be so divided as to read areas directly.

Let one end of AB (Fig. 7) be constrained to move along a branch AC of any curve, while the other end passes once around a closed line D. Any area outside of D which is swept at all must be swept as many times to the right as to the left, and all parts of D are swept once more to the right than to the left, so that the total area swept by the line is equal to the area of D. In its simplest form the planimeter consists of an arm AB with a rolling wheel. The end A is constrained to move on the arc of a circle, being hinged to one end of an auxiliary arm, the other end of which is fixed by a pivot.

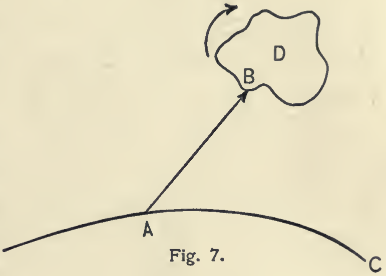

Fig. 7.

7. Mass. — The *Kilogram* is defined as the mass of a piece of platinum deposited in the Archives in Paris. The gram, which has been adopted by common agreement as the fundamental unit of mass, is one-thousandth of a kilogram. This kilogram was made by Borda to be equal to the mass of 1000 c.c. of pure water at 4° C., but copies of the kilogram can be made with much greater accuracy and with greater convenience by direct comparison with the "Kilogramme des Archives" than by comparison with the cubic decimeter of water, so that the above has been adopted as the definition.

The Pound (avoirdupois), which is the British commercial unit of mass, is the mass of a piece of platinum deposited in London. The pound (avoirdupois) is equal to 0.453593 kilograms.

8. Measurement of mass: the balance. — The masses of two bodies are considered equal when they are attracted by the earth with equal force. The *analytical balance*, the essential parts of which are shown in Fig. 8, is the most sensitive instrument for the determination of mass, and indeed the most precise of all the apparatus used in physics. It is a very delicately mounted equal-arm lever with suspended pans.

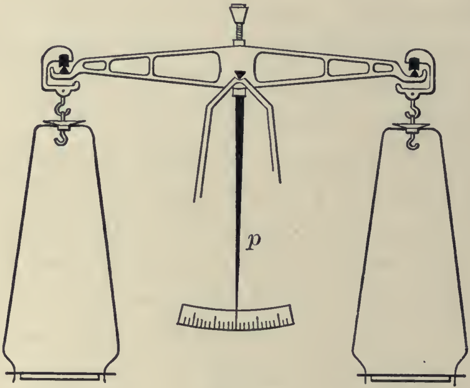

Fig. 8.

The balance is used for indicating the equality of the masses of two bodies. It is evident that in making a measurement the position of rest of the pointer *p* with loaded pans must be the same as with empty pans if the masses compared are equal. The masses are not generally exactly equal, however, and the slight difference between them is determined by comparing the difference of these two positions of rest of the pointer, with the shift produced when a small known mass is added to one pan.

In practice, these positions of rest are determined by starting the balance swinging and observing an odd number of elongations of the pointer. The position of rest lies midway between the mean of the elongations to the right and the mean of the elongations to the left. This is called *weighing by swings*.

9. **Weights.** — *A set of weights*, by means of which the mass of any body may be matched, is made by taking two bodies weighing together one kilogram and equalizing them, then two bodies weighing together half a kilogram and equalizing them, and so on. A set of weights more convenient in use is one which includes a five, a two, and two ones of each units, tens, hundreds, etc., of grams.

The indications of a balance are subject to error because of (*a*) inequality of arms, (*b*) difference in buoyant force of the air upon the body and upon the weights, and (*c*) errors of weights.

Errors due to inequality of arms are eliminated by weighing the body first on one pan and then on the other. As the result of this operation we have $b = \sqrt{W_r W_l}$, in which b is the true mass of the body, and W_r and W_l are the apparent masses when weighed upon the right and left pans respectively.

Proof. — Since r and l are the lengths of the right and left arms of the balance respectively, we have from the principle of the lever $W_r l = br$ and $W_l r = bl$; from which by elimination of $\frac{r}{l}$ we have $b = \sqrt{W_r W_l}$.

The errors due to the buoyant force of air are easily eliminated when the density, δ, of the weights; the density, Δ, of the body; and the density, λ, of the air, are known. Let b be the apparent mass of the body and W its mass independently of errors due to buoyant force of air. Then

$$W = b - \frac{b\lambda}{\delta} + \frac{b\lambda}{\Delta}. \tag{4}$$

In this expression, $\frac{b}{\delta}$ is the volume of the weights, and $\frac{b\lambda}{\delta}$ is the buoyant force of the air upon the weights, by which amount the mass of the body is overestimated. Similarly, $\frac{b\lambda}{\Delta}$ is the buoyant force of the air upon the body by which amount the mass of the body is underestimated.*

10. Errors of weights.—Consider a set of weights A, B, C, D, and E of nominal values 500 gr., 200 gr., 100 gr., 100 gr., and 100 gr., respectively. Having a small weight of which the mass is known approximately, we weigh A in terms of B, C, D, and E, whence

$$A = B + C + D + E + a; \tag{5}$$

similarly by repeated weighing we obtain,

$$B = C + D + b, \tag{6}$$

$$C = D + c, \tag{7}$$

$$D = E + d. \tag{8}$$

Having a standard kilogram, we weigh it also; whence we obtain

$$1000 = A + B + C + D + E + e. \tag{9}$$

From these five equations, A, B, C, D, and E, which are the only unknown quantities, may be calculated.

11. Density. — The *density* of a body is defined as its mass per unit volume. This definition may be expressed by the equation

$$D = \frac{M}{V}; \tag{10}$$

in which D is the density, M is the mass, and V is the volume of the body.

The *specific gravity* of a substance at a given temperature is

$$S = \frac{D}{d}, \tag{11}$$

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* For further discussion of the theory of the balance, and for descriptions of various forms of balances and scales, see Violle, Cours de Physique, I. p. 245; also Encyclopedia Britannica, 9th ed., art. *Balance.*

in which S is the specific gravity of the substance and $\frac{D}{d}$ is the ratio of its density D at that temperature to the density d of water at the same temperature.

Since masses of equal volumes of a substance and of water are proportional to their densities, we have also

$$S=\frac{M}{W}, \tag{12}$$

in which M is the mass of a portion of the substance, and W is the mass of the same volume of water.

12. Measurement of density. — The *specific gravity* of a substance is generally determined by weighing equal volumes of the substance and of water. For this operation the principle (Archimedes) that the buoyant force of a liquid upon a submerged body is equal to the weight of a volume of the liquid equal to the volume of the body, furnishes the most feasible and accurate method. To obtain the density, equation (11) is then used.

The use of this equation involves a knowledge of the density of water, which has been determined with great accuracy at a standard temperature by finding the buoyant action of the liquid upon an accurately ground cubical block of glass of which the dimensions have been measured and the volume calculated (Borda). The density of water at other temperatures has been determined by methods, the discussion of one of which is given in article 232.

13. Gravimetric methods for measuring volume.—The volume of any solid may be calculated from equation (10), the mass of an equal volume of water having been determined and the density of water being known. Similarly, the volume of a vessel may be found, the mass of water, or mercury, held by the vessel having been determined. For measuring liquids, when little accuracy is required, graduates are used.

A *graduate* is a vessel, best cylindrical, upon which a scale has been constructed by marking the position of the surface of successive additions of mercury, or of water, of known volume. Such a vessel is useful for quick measurements.

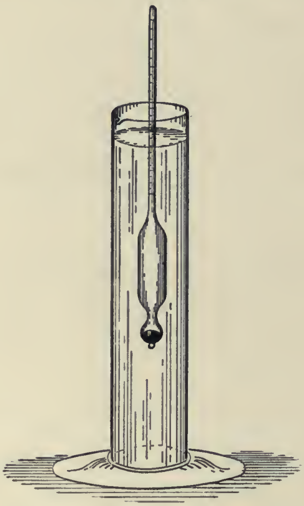

Fig. 9.

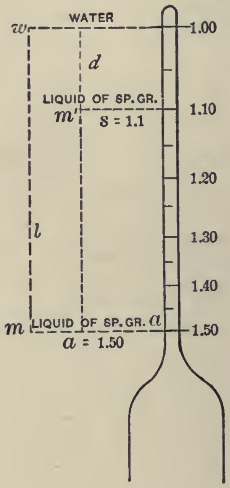

Fig. 10.

14. The hydrometer. — This is an apparatus much used for the determination of the density of liquids. It is a glass float (Fig. 9) with a vertical cylindrical stem, upon which is a scale for indicating specific gravity of any liquid in which it is placed.

Hydrometers were originally constructed with various arbitrary scales, of which that of Beaumé is the best known. It is, indeed, still sometimes used in commercial work. Modern hydrometers, however, are generally graduated so as to give the specific gravity directly, or to indicate the percentage strengths of the particular solution, in the testing of which the hydrometer in question is to be used.

For the construction of a *specific gravity* scale, the water-mark w (Fig. 10), and the mark m, to which the instrument sinks in a liquid of known specific gravity a, are determined, and the distance l between these marks is measured. The distance d from the water-mark to the mark m', to which the instrument will sink in a liquid whose specific gravity is s, is

$$d = l\frac{1 - \frac{1}{s}}{1 - \frac{1}{a}}; \tag{13}$$

from which the position of each mark of the scale may be calculated. The figure shows such a scale for liquids between sp. gr. 1.00 and sp. gr. 1.50.

Proof. — A floating body displaces its weight of liquid. The volume of water displaced by the instrument being considered unity, the displaced volume of the liquid of sp. gr. a is $\frac{1}{a}$, and the displaced volume of a liquid of sp. gr. s is $\frac{1}{s}$; also the volume of the stem, between water-mark and a-mark is $1 - \frac{1}{a}$, and the volume of stem between water-mark and s-mark is $1 - \frac{1}{s}$. Since the stem is cylindrical, these volumes are directly proportional to the distances l and d; whence the formula.*

As has already been stated, hydrometer scales are frequently constructed to give directly percentage strengths of solution of a given substance. To this end specific gravities of solutions of various percentage strengths are determined, and these specific gravities are substituted for s in the above formula, and the marks thus found are numbered as percentages.

A class of instruments depending upon a slightly different principle are the *hydrometers of constant immersion* of Fahrenheit (Fig. 11 *a*) and of Nicholson (Fig. 11 *b*). Both of these may be used for comparing the densities of liquids by determining the weights which must be placed upon the scale pan (s_1) in order to submerge the hydrometer to a certain mark upon the

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*The true sp. gr. corresponding to any reading on the Beaumé scale may be calculated from this formula, putting, for scale of heavy liquids, $l = 15$, $a = 1.110$, which is the sp. gr. of a 15 % solution of NaCl; s is then the true sp. gr. corresponding to d° Beaumé.

stem. Nicholson's hydrometer, which carries a scale pan (s_2) at the bottom also, is likewise used as an ingenious sort of *water* balance, for determining the specific gravities of solids by weighing in air and in water.

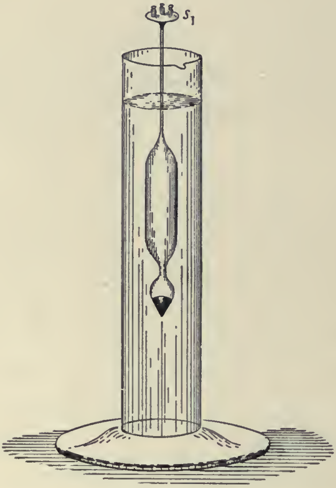

Fig. 11 (*a*).

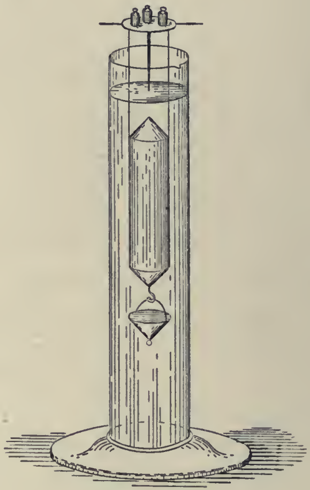

Fig. 11 (*b*).

15. Time. — The *second,* which is the accepted unit of time in physical measurements, is defined as the 86400th part of a mean solar day.

Measurement of time. — Any movement of a body which repeats itself in equal intervals of time is called *periodic motion;* single movements are called *vibrations.* All methods, with unimportant exceptions, for measuring time depend upon periodic motion. A vibrating *pendulum* is the most familiar example. The number, a, of vibrations in a day, and the number, b, in the interval to be measured, are counted. The interval is then $\frac{b}{a}$ of a day. A *clock* is simply a machine for maintaining and counting the vibrations of a pendulum. In portable clocks a "*balance wheel*" takes the place of a pendulum.

Anything which affects the time of vibration of a pendulum leads to erroneous values for time intervals as measured by a clock. The time of vibration of a pendulum is affected (*a*) by temperature, on account of increase of length of pendulum with temperature; (*b*) by variations of atmospheric pressure, on account, mainly, of variation of buoyant force of air with pressure; (*c*) by variation in amplitude of vibration; (*d*) by irregularities in the manner in which impulses are imparted to the pendulum by the clockwork. The influence of temperature is avoided by using what are called *compensated* pendulums, which do not change their effective length with temperature. The variations due to (*c*) are in part obviated by providing constant driving power, which requires the gears and escapement to be of fine workmanship. The variations due to (*d*) are obviated by using what are called *dead beat* escapements, which impart their impulse to the pendulum at the instant when it passes through the vertical position.*

16. The chronograph. — The determination of a time interval by means of a clock requires the clock reading to be taken at the beginning and at the end of the interval. Practice enables an observer to take the clock reading at the instant of a given signal accurately to a tenth of a second, with an approximately constant "personal" error, which does not greatly affect the value of the interval. In this method, which is called the *eye and ear* method, the observer looks for the signal and listens to the beats of the clock.

The chronograph (Fig. 12) is an instrument for enabling clock readings to be taken with greater ease and accuracy than is possible by "eye and ear." It consists essentially of a uniformly moving strip of paper usually wrapped about a cylinder. Upon this a line is traced by a pen. This pen is fixed to the

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* For discussion of errors of clocks, and description of escapements, see Encyclopedia Britannica, 9th ed., article *Clock*. For description of compensated chronometer balance and chronometer escapement, see Lockyer's book entitled *Star-gazing*, pp. 175 to 210.

armature of an electro-magnet, which is excited at each beat of the clock by means of an electrical contact device actuated by the pendulum. A kink is made in the traced line at each excitation of the magnet. At the instant for which the clock reading is desired, the electro-magnet is momentarily excited by pressing a key which closes an auxiliary circuit. The clock reading is determined by measuring off the position of the kink thus produced, among the kinks produced by the beats of the pendulum.

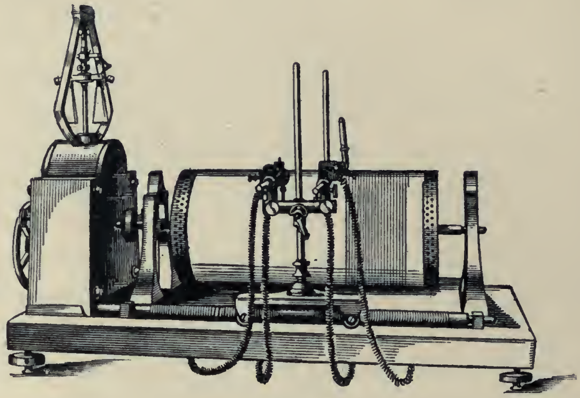

Fig. 12.

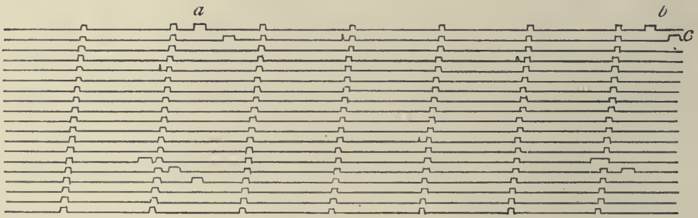

Fig. 13.

Figure 13 shows the reduced facsimile of a portion of the sheet upon which a chronographic record has been made. The kinks marked *a*, *b*, *c*, etc., are those by means of which the observations were recorded.

CHAPTER II.

PHYSICAL QUANTITY.

17. In the expression of a physical quantity two factors always occur, a numerical factor and a unit. The numerical factor is called the *measure* of the quantity and its determination is the object of physical measurement.

Algebraic operations upon physical quantity involve both *unit* and *measure;* for example, $am \times bn = ab \cdot mn$, and $am \div bn = \frac{a}{b} \cdot \frac{m}{n}$, where a and b are numerical factors, and m and n are physical units. In such cases mn and $\frac{m}{n}$ are considered new physical units. Thus 10 c.m. × 10 c.m. $= 100\ (\text{c.m.})^2$; 20 grams ÷ 10 c.c. $= 2\frac{\text{gr.}}{\text{cc.}}$, read *grams per cubic centimeter.* The word *per* indicates that the unit following is in the denominator.

18. Units, fundamental and derived. — The *fundamental* physical units are those fixed by arbitrary, preserved standards.

There is much latitude in the choice of fundamental units. A single fundamental unit is sufficient, but to derive all physical units from one would lead to many cases in which the measurements for the determination of a copy of a derived unit would be much less accurate than the measurements in which this copy is used as a working standard. The units of *length*, *mass*, and *time* have been adopted as fundamental units.

Derived physical units are those which are defined in terms of the fundamental units and of which no material standard need be preserved. It often occurs that copies of a derived unit are carefully made and preserved, and used as "working standards" of reference.

THE C. G. S. SYSTEM OF UNITS.

Derived units based upon the *centimeter* as the unit length, the *gram* as the unit mass, and the *second* as the unit of time, are in common use. This system of derived units is called the c. g. s. system.

19. Dimensions of derived units. — The definition of a derived unit in every case implies an equation which involves that unit together with the units of length; mass, and time. This equation solved for the derived unit is said to express the *dimensions* of that unit. For example, let l be a unit length, m a unit mass, and t a unit time; then area $=l^2$, volume $=l^3$, density $=\frac{m}{l^3}$, velocity $=\frac{l}{t}$, force $=\frac{ml}{t^2}$, magnetic pole $=\frac{m^{\frac{1}{2}}l^{\frac{3}{2}}}{t}$, etc.

In those cases in which a derived unit of the c. g. s. system has no specific name it is specified by its algebraic expression in terms of the centimeter, the gram, and the second. Thus the unit of momentum is written $\frac{\text{gr. c.m.}}{\text{sec.}}$ (spoken *gram-centimeter-per-second*), and the unit of moment of inertia is written gr. cm.^2 (spoken *gram-centimeter-square*). In case of units which have complicated dimensions this method is not convenient in speech. Thus we specify a certain magnetic pole as $150\,\frac{\text{gr.}^{\frac{1}{2}}\text{cm.}^{\frac{3}{2}}}{\text{sec.}}$, spoken 150 *c. g. s. units pole.* Many derived units have received specific names. Such are the *dyne*, the *erg*, the *ohm*, the *volt*, etc.

Secondary derived units. — In many cases the c. g. s. unit of a quantity is either very large or very small in comparison with ordinarily occurring values of the quantity. It is then more convenient to use a submultiple or a multiple of the c. g. s. unit. Such units are called *secondary* units or *practical* units. For example, an ordinary arc lamp has 5,000,000,000 c. g. s. units, or 5 *ohms* of electrical resistance. An ordinary Leyden jar has, say, 5×10^{-18} c. g. s. units, or .005 micro-farads of electrostatic capacity. It is convenient and customary in writing large numbers to use some positive integral power of ten as a factor, and, in writing very small numbers, to use, as above, some negative integral power of ten as a factor.

20. Scalar and vector quantities. — A *Scalar* is a quantity which has magnitude only. Volume, mass, time, energy, electric charge, etc., are scalars. Scalars conform to all the ordinary rules of algebra; indeed, ordinary algebra is the algebra of scalar quantity.

A *Vector* is a quantity which has magnitude and direction. Length, area, velocity, force, etc., are vectors. A vector requires three independent scalar specifications, which are called the *elements* of the vector. These elements may be, for example, the components of the vector in each of three chosen directions.

A vector equation, $\alpha = \beta$, is equivalent to three scalar equations, for the scalar elements of α must be equal to the three scalar elements of β, each to each.

Vector algebra differs from scalar algebra, first, as to conceptions of addition, subtraction, multiplication, and division, and, second, as to rules of symbolic operations. This second point of difference grows largely out of the fact that in vector analysis $\alpha\beta = -\beta\alpha$, and although it is important in mathematical physics, it cannot be discussed here.

The first point of difference is more intimately connected with the physical nature of vectors, and is exceedingly important to the beginner. All vector quantities grow out of *Length*, and depend either upon single lines or upon the mutual relation of two or more lines, so that the principles of vector algebra, when established for the *line*, are completely established, and need no further proof. It is true, moreover, that vectors of any kind are best *represented* by lines, to scale. Hereafter letters will be occasionally used for vectors and scalars indiscriminately, and the student will be obliged to *bear in mind* the distinction between the two.

21. Vector addition. — A number of similar vectors are represented by lines α, β, γ (Fig. 14); line α being drawn from a point o, line β from the extremity of α, line γ from the extremity of β, etc. The line drawn from o to the terminus of the last line represents the sum of the vectors.

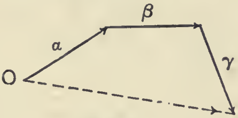

Fig. 14.

Proof. — The vectors α, β, γ, etc., being linear displacements of a point, their sum is understood to mean the single displacement equivalent to them all, and this is, of course, the single displacement from the initial to the final position of the point.

The vector polygon. — The sum of a number of similar vectors is zero when they are parallel to and proportional to the respective sides of a closed polygon, and in the directions in which these sides would be traversed in going round the polygon.

22. Resolution of vectors. — Any vector may be replaced by a number of vectors of which it is the sum. The simplest case is that in which a vector is replaced by two vectors which are parallel and proportional to the respective sides of a parallelogram, of which the diagonal represents the given vector. If a rectangular parallelopiped be constructed whose diagonal represents a given vector, then the edges of the parallelopiped will represent what are called the *rectangular components* of that vector in the directions of those edges.

23. Scalar and vector products and quotients. — The product or quotient of a scalar and a vector is another vector parallel to the vector factor.

Examples. — (1) The expression $F=pa$, in which p is an hydrostatic pressure, a an exposed plane area, and F is the force acting upon that area. (Note that the vector direction of an area is its normal.)

(2) The expression $l=vt$, in which v is the velocity of a moving body, and l is the distance traveled in time t.

(3) The expression $f=\frac{F}{m}$, in which F is the force with which a magnetic pole m is acted upon in a magnetic field of strength f.

(4) The expression $\frac{W}{d}=F$, in which W is the work done upon a body in moving it through a distance d, and F is the force parallel to d, which must be acting upon the body.

24. Vector Products.

Case I. *Parallel vectors.* — The product or quotient of parallel vectors is a scalar. Thus $W=Fd$, in which W is the work done by a force F acting through a distance d in its direction; $p=\frac{F}{a}$, in which p is the pressure in a liquid which exerts a force F on an exposed area a; $V=la$, in which V is the volume of a prism of base a and altitude l.

Case II. *Orthogonal vectors.* — The product of two mutually perpendicular vectors is a third vector at right angles to both factors. Thus $a=lb$ and $b=\frac{a}{l}$, in which a is the area of a rectangle of length l and breadth b; $T=Fl$, in which T is the *moment* or *torque* of a force F, and l is its arm. The product of a vector and a line perpendicular thereto is called a *moment* of the vector.

Case III. *Oblique vectors.* — The product of two oblique vectors consists of two parts, one of which is a scalar and the other is a vector. Consider two vectors, α and β (Fig. 15). Resolve β into two components, β' and β'', respectively parallel to and perpendicular to α. Then

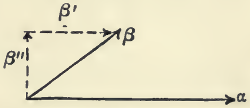

Fig. 15.

$$\alpha\beta=\alpha(\beta'+\beta'')=\alpha\beta'+\alpha\beta'',$$

in which $\alpha\beta'$ is a scalar and $\alpha\beta''$ is a vector. The scalar part of a vector product is indicated thus, $S.\,\alpha\beta$ (read *scalar-alpha-beta*).

The vector part of a vector product is indicated thus, $V.\ \alpha\beta$ (read *vector-alpha-beta*). When $V.\ \alpha\beta = 0$, α and β are parallel; when $S.\ \alpha\beta = 0$, α and β are orthogonal.

Examples. — (1) $V.\ bl$ is the area of any parallelogram of which b and l are the sides.

(2) $S.\ al$ is the volume of any parallelopiped of which a is the area of the base and l is the other edge.

25. Constant and variable quantities. — Many physical quantities are either constant or they change so slowly that the phenomena which depend upon their variation are inconsiderable. In the study of phenomena which depend upon such quantities, the attention need not be directed to what is taking place at a particular instant, inasmuch as the phenomena persist unchanged throughout long intervals of time.

Other physical quantities are peculiarly subject to rapid variations, and it often occurs that certain phenomena depend upon these variations in such a way as to be inconsiderable or non-existent when these changes take place slowly or not at all. Thus all kinds of force, including torque, pressure, electromotive force, etc., are intimately associated with changing states of motion and would no doubt be absolutely non-existent if the state of motion of the involved bodies and particles were strictly constant. In the study of such phenomena the attention must be directed to what is taking place at a particular instant.

The *average rate of change* of a quantity x during a time interval Δt is defined as the quotient obtained by dividing the change Δx suffered by that quantity during the interval, by the interval. The *instantaneous rate of change* of x is the limiting value of this quotient when the time interval is very small. The instantaneous rate of change of x is sometimes represented by the symbol $\dot{x}$, sometimes by the symbol $\frac{dx}{dt}$. In case $\dot{x}$ is variable, its rate of change is defined in the same way, and is called the *second rate of change* of x. It is represented by $\ddot{x}$ or

$\frac{d^2x}{dt^2}$. The dimensions of $\dot{x}$ and $\ddot{x}$ are $\frac{x}{t}$ and $\frac{x}{t^2}$ respectively. If x is a scalar, $\dot{x}$, $\ddot{x}$, etc., are scalars, since Δx is then a scalar and Δt is a scalar. If x is a vector, then $\dot{x}$, $\ddot{x}$, etc., are vectors. The direction of $\dot{x}\left(=\frac{\Delta x}{\Delta t}\right)$ is that of Δx, see Art. 23.

Consider a changing vector α, and let it be represented at each instant by a changing line l (Fig. 16) drawn to scale, from a stationary point O. Then $\alpha = s\,l$, where s is the *scale number* of the diagram. Let Δl be the change in l during Δt; then

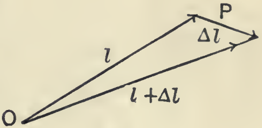

Fig. 16.

$$\Delta\alpha = s\,\Delta l \text{ and } \frac{\Delta\alpha}{\Delta t} = s\,\frac{\Delta l}{\Delta t}.$$

Now the limiting value of $\frac{\Delta\alpha}{\Delta t}$ is $\dot{\alpha}$, and the limiting value of $\frac{\Delta l}{\Delta t}$ is the *velocity* of the point P, therefore *if any vector is represented to scale by a line OP drawn from a stationary point, then the rate of change of that vector is at each instant represented to the same scale by the velocity of the point P.*

26. Distributed quantity. — A physical quantity may refer to the state of a medium, having definite values at each point, in such a way that the value of the quantity at a point specifies the state of the medium at that point. Such a quantity is called a *distributed quantity.* Thus in a liquid the pressure has at each point a definite value (distributed scalar), and a moving liquid has at each point a definite velocity (distributed vector). A magnetic field, in the same way, has at each point a definite intensity (distributed vector). Another example is that of the *strain* at each point in a distorted solid, which is definite in value.

The distribution of a quantity is said to be *homogeneous* or *uniform* when it has the same value at all points; otherwise the distribution is said to be *non-homogeneous.* In the study of phenomena, dependent upon distributed quantity, in cases in which the distribution is non-homogeneous, the attention must be directed to what takes place in small regions.

27. Complex physical quantity. — A scalar may be defined as a quantity which depends upon a single numerical specification. A vector, as has already been indicated, depends upon three independent numerical (scalar) specifications, and is distinguished among others of the same degree of complexity by the interpretation adopted for products and quotients of the scalar units. There are quantities, however, which are more complex. Thus *stress* and *strain*, in the simplest case, depend each upon six independent numerical (scalar) specifications, and there are other physical quantities of still higher complexity. All such complex quantities are intimately connected with geometry, and it is very important that the student attain to a degree of familiarity with them, inasmuch as the imaging faculty is greatly strengthened thereby. For purposes of numerical calculation scalars only are important. For such purposes all complex quantity must be degraded to scalars, and the scheme made use of in this degradation must be kept in mind. Such schemes are the essence of physical theory. Methods for reaching numerical results by vector calculations can only be attained by the use of geometrical-mechanical appliances. Arithmetical methods are essentially scalar, and require all data to be reduced to that character.

CHAPTER III.

LAWS OF MOTION; FALLING BODIES; PROJECTILES.

28. In extended treatises the subject of mechanics is usually divided into **kinematics,** which is properly a branch of geometry and treats of motion in the abstract; and **dynamics,** which treats of the motion of bodies under the action of force.

Dynamics is itself divided into **statics** and **kinetics.** Statics is the study of forces in equilibrium, and kinetics is the study of the motion of bodies upon which unbalanced forces act. For our present purpose it is not convenient to treat these different branches separately.

29. A **material particle** is a body of such small dimensions that it may be considered to be concentrated at a point. The earth, for example, as a part of the solar system, may be treated as a particle. Any rigid body which is constrained to move without rotation may be treated as a particle so far as concerns its behavior under the action of force.

30. The position of a particle is specified by its co-ordinates x, y, and z referred to arbitrary axes of reference. The line drawn from the origin of co-ordinates to the particle is called its **position vector.** This line is always to be thought of when we speak of the position of a particle.

31. The displacement of a particle is a change in its position. The displacement is specified by the changes, Δx, Δy, and Δz, in the co-ordinates of the particle. The line drawn from the initial to the final position of a displaced particle is called its **displace-**

ment vector. This line is always to be thought of when we speak of the displacement of a particle.

32. The addition of displacements. — Any number of consecutive displacements *a*, *b*, *c*, and *d* of a particle *P* (Fig. 17), are together equivalent to the single displacement *e*, which is their vector sum. Displacement is the fundamental vector quantity, and the proof of the addition theorem for displacements involves the proof of that theorem for all vector quantities. (See Art. 21.)

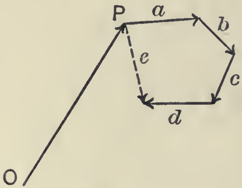

Fig. 17.

33. The velocity v of a particle is the rate of change of its position; that is, it is the limiting value of $\frac{\Delta l}{\Delta t}$, in which Δl is the displacement of the particle during the interval Δt. The components of v are $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$. The dimensions of velocity are $\frac{l}{t}$. The unit velocity, one centimeter per second, is sometimes called the *kin* for brevity.

The orbit of a particle is the path it describes in its motion. The velocity of the particle is at each instant tangential to the orbit.

The acceleration $\dot{v}$ of a particle is the *rate of change* of its velocity. The components of $\dot{v}$ are $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$. The dimensions of $\dot{v}$ are $\frac{l}{t^2}$.

34. Graphical representation of velocity and acceleration. — Let the line l (Fig. 18), drawn from a fixed point O', represent the velocity of a moving particle at a given instant. Then $v = sl$, s being the *scale* to which v is represented by l.

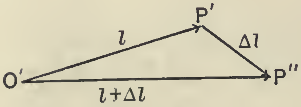

Fig. 18.

After a time Δt has elapsed the velocity becomes

$$v+\Delta v=s(l+\Delta l),$$

so that $$\Delta v=s\Delta l \text{ or } \frac{\Delta v}{\Delta t}=s\frac{\Delta l}{\Delta t} \text{ or } \dot{v}=s\frac{\Delta l}{\Delta t};$$

but $\frac{\Delta l}{\Delta t}$ is the velocity of the point P', hence, *if the velocity of a moving particle be represented, to scale, by a line drawn from a fixed point, the* **velocity** *of the moving end of the line represents the acceleration of the particle to the same scale.*

35. The hodograph to an orbit is the curve traced by the point P' in Fig. 18, the line $O'P'$ being drawn so as to represent the velocity of the moving particle at each instant. The point P' is called the generating point of the hodograph.

36. Components of acceleration perpendicular to and parallel to velocity.—The acceleration of a particle may be resolved into two components at each instant, one parallel to v and the other perpendicular thereto. The former component affects only the magnitude of v, while the latter affects the direction of v and determines the curvature of the orbit, as follows:

Let r be the radius of curvature of the orbit, v the velocity of the particle, and $\dot{v}$ the component of the acceleration perpendicular to v; then (article 58, equation 34):

$$r=\frac{v^2}{\dot{v}}. \qquad (14)$$

37. Newton's Laws of Motion.

I. All bodies persevere in a state of rest or in a state of uniform motion in a straight line, except in so far as they are made to change that state by the action of force.

II. (*a*) The rate at which the velocity of a particle changes is parallel and proportional to the force acting upon the particle.

(*b*) The rate at which a given force changes the velocity of a particle is inversely proportional to the mass of the particle.

III. Action is equal to reaction, and in a contrary direction.

The first law describes the behavior of a body upon which no force acts. When the vector sum of a number of forces acting upon a particle is zero, the forces are said to be in equilibrium and the particle behaves as if no force were acting upon it. Any body, as a boat or a train, which, even for a short time, moves uniformly in a straight line, is a realization of the conditions of the first law.

The second law describes the behavior of a body when acted upon by a force, and is really the definition of force. The third law states that all forces are due to mutual action of bodies, and that the mutual action of two bodies consists of two equal and opposite forces, which act, one on each body, so that a change in the state of motion of a body is always associated with a simultaneous definite change in the state of motion of some other body. All physical phenomena are ultimately connected with such correlated changes of motion. The mutual force action of two bodies is a *stress*, in an intervening medium or in a connecting mechanism.

38. From the second law it is evident that the force which acts upon a particle is proportional to the product of the mass of the particle and its acceleration; hence, if the unit force be chosen as that force which produces an acceleration of *one kin per second* (Art. 39) in a particle of which the mass is one gram, then

$$F = m\dot{v}. \tag{15}$$

In this equation m is the mass of a particle and $\dot{v}$ is the acceleration produced by a force F.

The unit force here defined is called the **dyne.**

The dimensions of force are $\frac{ml}{t^2}$.

39. Illustrations of the Laws of Motion.

Consider a string from which a bit of metal is suspended; it is in tension, so that a force acts on the bit of metal, which force balances the attraction of the earth for the metal (see Arts. 40 and 41). Take the string in hand and give the bit of metal a motion of rotation. The bit of metal then moves with a continually varying velocity (the reader must not think only of the magnitude of velocity which is a vector). The acceleration, $\dot{v}$, of the bit of metal is in the direction of the string (see Art. 58), and may be very great. The string must be pulling on the bit of metal with a force which is proportional to $\dot{v}$ by the second law. The bit of metal reacts and pulls on the string and, through the string, on the hand. A familiar example is afforded in the operation of casting from the shore with the hand line, as practised in the taking of bluefish. The fisherman swings his leaded hook around his head. The mass of the hook is, perhaps, three or four hundred grams, but the force upon the line, necessary to constrain the hook to a circular path (second law) soon becomes almost great enough to throw the fisherman from his feet. When released, the hook travels, perhaps, a hundred meters, carrying after it several hundred grams of line, and falls into the water beyond the breakers.

A force is necessary to constrain each separate particle of a rotating body to move in a circle, and it is a well-recognized fact in the construction of machinery that no wheel, however strong, can be driven beyond a certain speed without rupture.

In the case of a railway train on a curve, a side force must be applied to change the direction of its motion. This side force, being proportional to the acceleration, is proportional to the square of the velocity of the train and inversely proportional to the radius of the curve (see Art. 58, Equation 33). Ultimately it becomes so great that it cannot be applied with safety, so that one of the requisites of the very highest attainable speed upon railways is the abolition of curves.

When a body, once put into motion, comes gradually to rest, as most bodies do unless some properly directed force-action upon them from without be maintained, the checking of the velocity, like the production of it, is due to the action of some force.

The third law of motion is really a terse statement of the nature of force. So long as the intervening medium by means of which force is transmitted is a solid, the truth of the law is obvious at once. It is obvious, for example, that the pressure which can be exerted by means of a screw press is limited by the resistance which the nut and its fastenings can offer to the back thrust of the screw; likewise that the structure which prevents the shaft of a steamship from being driven forward through the vessel must be capable of withstanding a force equal to that by means of which the latter is urged through the water.

It is important to recognize that the law is a perfectly general one; in other words, that the force-ties between bodies are of the same nature when the medium is a liquid or gas, or even when it is the imponderable material which fills the interstellar spaces.

In the case of liquids and gases there are numerous familiar phenomena by means of which the third law may be illustrated.

The phenomena of the reaction of liquid jets is utilized in a variety of ways. It has been found feasible, for example, although not economical, to propel vessels by means of steam pumps which draw in water through pipes at the bows and expel it at the stern. This plan is analogous to the method of towing upon some European rivers which are too swift for ordinary navigation, and where boats draw themselves up stream by the continuous grappling of a cable which lies in the river bed.

40. Weight. — It is found that at a given place on the earth the acceleration g of a freely falling body is constant. This constant acceleration is called the intensity of gravity at the place. The force W which gravity exerts on a body is called the *weight* of the body. From equation (15) we have

$$W = mg, \tag{16}$$

in which W is the weight of a body in dynes, m is its mass in grams, and g is the acceleration of gravity in kins per second.

41. Gravitational units of force. — The intensity of gravity is subject to but slight variations at different parts of the earth's surface, so that the weight of a pound, or the weight of a gram, is a convenient unit of force for some practical purposes. The weight of a pound, when employed as a unit of force, is called, for brevity, **a pound**.

42. Measurement of Force.

(*a*) *Kinetic method.* — The force acting upon a body of known mass may be calculated by equation (15), the acceleration of the body being determined by observation. This method cannot be realized in its simplicity, but it forms the basis of many physical measurements.

(*b*) *Counterpoise method.* — The force to be measured is applied at one end of a lever, and counterpoised by weights of known mass hung on the other end. The magnitude of the force is then easily calculated in gravitational measure. In case the force to be measured is the breaking force of a wire or beam, or the force necessary to produce a given elongation or flexure, it may be applied directly by means of weights of known mass.

(*c*) *By means of the spring dynamometer.* — The elongation of a spring is proportional to the stretching force. The elongation due to a known force being observed, the force which produces any observed elongation may be calculated.

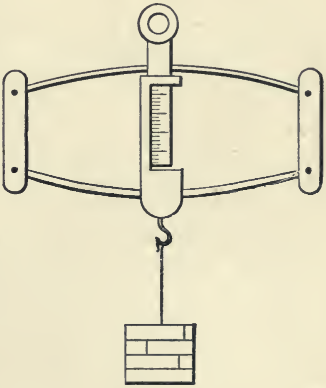

Fig. 19.

Frequently, where the forces to be measured are large, bent springs are used, with a device for magnifying the movement.

Figure 19 shows a well-known form. Such dynamometers are calibrated by applying a series of known forces and marking the deflections which they produce.

43. System of particles. — A number of particles of which the aggregate motion is the object of study is called a system of particles; a portion of water, for example, is a system of particles. The **collective positions** of the particles of a system is called the **configuration** of the system. The **collective velocities** of the particles of a system is called the **velocity configuration** of the system. The expression "state of motion" of a system will be used to signify both the **position configuration** and the **velocity configuration** taken together.

44. A closed system is one which has no force acting on any part of it from outside the system, or it is a system which neither gives energy to nor receives energy from any other system (see Art. 83).

45. The mass-vector of a particle is defined as the product of the mass of the particle into its position vector; that is,

$$M = ml, \tag{17}$$

in which M is the mass-vector of a particle, m is its mass, and l its position vector. The components of M are mx, my, and mz.

46. Center of mass. — The sum of the mass-vectors of all the particles of a system divided by the total mass of the system is the position vector of a point called the *center of mass* or the *center of inertia** of the system. The co-ordinates X, Y, and Z of the center of mass of a system are

$$\left.\begin{aligned} X &= \frac{\Sigma mx}{\Sigma m}, \\ Y &= \frac{\Sigma my}{\Sigma m}, \\ Z &= \frac{\Sigma mz}{\Sigma m}, \end{aligned}\right\} \tag{18}$$

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* Sometimes called also *center of gravity*.

as is evident when we consider that the components of the mass-vector of a particle are mx, my, and mz, and that the sum of the x-components of a number of vectors is the x-component of their sum (see Art. 21). It follows that Σmx is the x-component of the sum of the mass-vectors of the particles, and $\frac{\Sigma mx}{\Sigma m}$ is the x-component of the position vector of the center of mass of the system or the x-co-ordinate of that point. If the origin of co-ordinates is at the center of mass of the system, then $X = Y = Z = 0$, and equations (18) become

$$\begin{aligned}\Sigma mx &= 0,\\ \Sigma my &= 0,\\ \Sigma mz &= 0.\end{aligned} \tag{19}$$

47. Momentum. — The momentum of a particle is defined as the product of its mass into its velocity. Since the mass of a particle is constant, we have from equation (17)

$F = m\dot{v}$ ✓

$$\frac{dM}{dt} = m\frac{dl}{dt} \text{ or } \frac{dM}{dt} = mv. \tag{20}$$

The momentum of a particle is therefore equal to the rate of change of its mass-vector. From (20) we have

$$\frac{d^2M}{dt^2} = m\frac{dv}{dt} \text{ or } \frac{d^2M}{dt^2} = m\dot{v}; \tag{21}$$

that is, the second rate of change of the mass-vector of a particle is equal to the force ($m\dot{v}$ by equation 15) which acts upon the particle.

48. Conservation of momentum. — Consider two particles of a system. Let m and m_1 be their masses, l and l_1 their position vectors, v and v_1 their velocities, and $\dot{v}$ and $\dot{v}_1$ the **parts** of their accelerations, which are due to their mutual action. From equation (15) and Newton's third law we have $m\dot{v} + m_1\dot{v}_1 = 0$, whence $mv + m_1v_1$ is a constant so far as the mutual action of these two particles is concerned. What is true of this pair of particles is true of every other pair in the system; hence *the sum of the momenta of the particles of a closed system never changes.*

49. Motion of the center of mass of a closed system. — The momentum of a particle is the rate of change of its mass-vector (see Art. 47); and by algebra

$$mv+m_1v_1+\cdots=\frac{d(ml)}{dt}+\frac{d(m_1l_1)}{dt}\cdots=\frac{d}{dt}(ml+m_1l_1+\cdots),$$

so that the sum of the momenta of the particles of a system is the rate of change of the sum of their mass-vectors.

Hence by Article 48 the sum of the mass-vectors of the particles of a closed system changes at a constant rate. From the definition of center of mass this requires the position vector of that point to change at a constant rate, and since the rate of change of the position vector of a point is the velocity of that point, it follows that *the center of mass of a closed system moves at uniform velocity in a straight line.*

If outside forces act upon the particles of a system producing accelerations $\dot{v}$, $\dot{v}_1$, $\dot{v}_2$, etc., then since $m\dot{v}=\frac{d^2(ml)}{dt^2}$ by Art. 47, we have

$$m\dot{v}+m_1\dot{v}_1+\cdots=\frac{d^2(ml)}{dt^2}+\frac{d^2(m_1l_1)}{dt^2}+\cdots$$

$$=\frac{d^2}{dt^2}(ml+m_1l_1+\cdots)=\Sigma m\cdot\frac{d^2L}{dt^2};$$

where $L=\frac{\Sigma ml}{\Sigma m}$, *i.e.* L is the position vector of the center of inertia of the system. But $m\dot{v}+m_1\dot{v}_1+\cdots$ is the vector sum F of the forces acting on the particles, and $\frac{d^2L}{dt^2}$ is the acceleration $\dot{V}$ of the center of inertia, so that $F=\Sigma m\cdot\dot{V}$. It follows that *the center of mass of any system of particles under the action of external force moves exactly as would a particle of mass* Σm

under the action of a force equal to the vector sum of the forces which act on the system.

50. Uniformly accelerated motion. — Consider a particle, a freely falling body for example, of which the velocity increases at the constant rate g. Let the x-axis of reference be chosen downwards in the direction of this constant acceleration; let v_1 be the velocity parallel to x at the instant from which time is reckoned, and let the origin of co-ordinates be chosen as the position of the particle at the same instant. When $t=0$, we have $\frac{dx}{dt}=v_1$, and $x=0$. The acceleration being parallel to x is $\frac{d^2x}{dt^2}$, so that

$$\frac{d^2x}{dt^2}=g; \tag{22}$$

whence, in general

$$\frac{dx}{dt}=gt+v_1, \tag{23}$$

and

$$x=\tfrac{1}{2}gt^2+v_1t. \tag{24}$$

If the particle have any horizontal velocity h, it will remain unchanged during the motion. Let the Y-axis be chosen in the direction of h; then the y-co-ordinate of the particle at a given time t is

$$y=ht. \tag{25}$$

Eliminating t from (24) and (25), we have

$$gy^2+2\,hv_1y=2\,h^2x, \tag{26}$$

which is the equation to the parabolic orbit of the particle.

51. Experimental study of the Law of Falling Bodies.

(1) Case of a body falling from a state of rest.

In equation (24) $v_1=0$,

and we may write $x=\frac{1}{2}gt^2$;

and when $t=1,\ x=\frac{1}{2}g$.

The relations to be verified by trial are:

$$x : t^2 \text{ and } x : g;$$

also

$$g = v_1 = 2\,x_1$$

(where v_1 and x_1 are velocity at end of one second and the path traversed during the first second respectively); and finally from equation (23)

$$v : t \text{ and } v : g.$$

The acceleration due to the gravity is so great that it is difficult to make direct observations upon a body falling freely through space under the full attractive force of the earth. It is possible, however, to reduce the force at will without materially changing the law of the falling body.

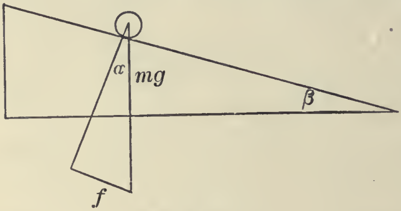


Fig. 20.

Galileo used for this purpose an inclined plane down which a ball was allowed to roll.

It is obvious that in this case the component of the earth's attraction (*f*, Fig. 20), which urges the ball down the plane, may be given any desired value, since

$$f = mg \sin \alpha = mg \sin \beta,$$

where β is the angle which the plane makes with the horizon.

The force is still a constant one, moreover, and the results obtained are applicable also to the case of a body falling freely through space.

By means of experiments made with this device, and checked by other methods, such as the observation of the time of flight of bodies allowed to fall freely from the various galleries of the leaning Tower of Pisa, Galileo established experimentally the relations contained in equations (23) and (24), and deduced the first and second laws of motion afterwards formulated by Newton.

A more modern, and, in some respects, a more refined device for the study of the laws of falling bodies, is *Atwood's Machine* (1784).

In this instrument two equal weights of mass M are suspended by means of a light cord from a perfectly balanced wheel (Fig. 21). A similar cord, *cc*, hangs from below the weights, so that the masses pendent at the two sides of the wheel will always be the same whatever the position of the weights may be. The wheel is mounted so as to be as nearly frictionless in its bearings as possible.

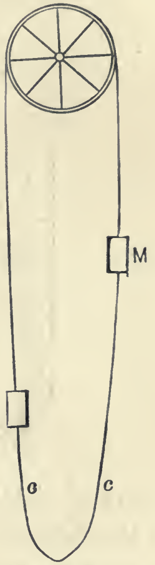


Fig. 21.

If a small additional weight m be added to either side, the mass on that side falls as though acted upon by a constant force $\frac{m}{2M+m+M_0}g$, where g is the acceleration of gravity and M_0 is a constant, called the equivalent mass of the wheel.

The machine is further provided with a vertical rod divided to millimeters, and carrying —

(*a*) at the top of the scale a hinged platform (*a*) (Fig. 22);

(*b*) an adjustable platform (*b*) (Fig. 23) sliding upon the rod by means of a collar or sleeve and capable of being set in any desired position;

(*c*) an adjustable ring (*r*) (Fig. 24), the aperture of which is sufficient to allow the weight M to pass.

The center of these platforms and of the ring coincide with the line of the suspension cord and the vertical axis of the suspended weight on that side of the machine.

There is also attached to the instrument a pendulum which beats seconds and which actuates a single mechanical or electrical device by means of which the hinged platform may be made to fall from under the weight $M+m$, previously resting thereon, at the beginning of a given second.

Of the small weights or riders, there are several, m_1, m_2, m_3, etc., differing in mass; each of them bearing a known and simple relation to that of the large weights *MM*. Some of these are of disk form with a slot (Fig. 25) (*a*) and capable of passing through the ring. Others are oblong (Fig. 25) (*b*) and will not pass through the ring.

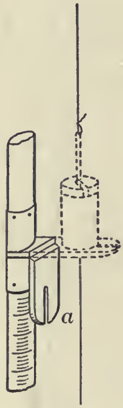


Fig. 22.

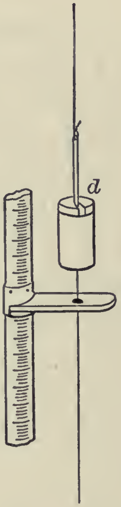


Fig. 23.

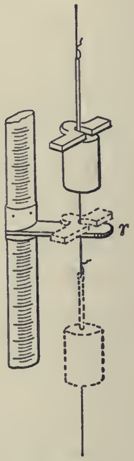


Fig. 24.

Fig. 25.

To verify the relation of distance to time, the weight *M*, carrying one of the riders, is placed upon the hinged platform. When released by the pendulum it falls with a uniformly accelerated motion. The positions upon the rod at which the adjustable platform must be placed receive the weight at the end of 1, 2, 3, etc.; seconds are determined by trial. Their distances x_1, x_2, x_3, etc., from the starting point are relatively as 1 : 4 : 9, etc., in conformity with the relation $x : t_2$, which is to be verified.

To verify the law of velocities, the oblong rider is used.

By placing the ring at a point on the rod such as to remove the rider at the end of the first second, the motion from that instant on is rendered uniform. The position of the platform (Fig. 23), when it checks the weight at the end of the subsequent second, is such as to demonstrate the relation

$$g = v_1 = 2x_1.$$

In a similar manner the relation of the velocity to the time may be studied.

To verify the relation x : f, where f is the acceleration due to the weight of the rider, two riders, the names of which are m_1 and m_2, are used, and the distances traversed under their action are noted.

In this case we have two equations,

$$\begin{aligned} f_1 &= g\frac{m_1}{2M + M_0 + m_1}, \\ f_2 &= g\frac{m_2}{2M + M_0 + m_2}, \end{aligned} \tag{27}$$

from which, by elimination of the common constants g and $(2M + M_0)$, we obtain the formula

$$\frac{f_1}{f_2} = \frac{m_1}{m_2}. \tag{28}$$

The experiment, therefore, is reduced to demonstrating the fact that the paths traversed in a given time are proportional to the masses of the riders.

52. The hodograph to the orbit of an uniformly accelerated particle is a straight line parallel to the acceleration. The velocity of the generating point P' is constant, and the ratio of this constant velocity of P' to the acceleration g of the particle is equal to the ratio of the distance $O'A$ to the horizontal velocity h of the particle; all of which is evident when we consider that radius vector $O'P'$ represents the velocity of the particle to the same scale that the velocity of P' represents the acceleration of the particle.

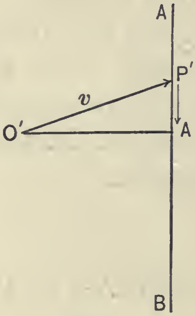


Fig. 26.

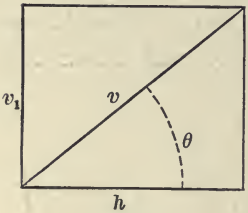


Fig. 27.

53. Projectiles. — Let v (Fig. 27) be the initial velocity of a projectile, and θ the angle of elevation of the gun; then $v \cos \theta = h$ and $-v \sin \theta = v_1$, where h and v_1 have the same significance as in article 50. The co-ordinates of the projectile after time t are given by equations (24) and (25).

The **range** of a gun is the horizontal distance traveled by the projectile by the time it has reached the level of the gun on its downward flight. The range l is found from equation (25) by putting $x = 0$ and writing the above values for h and v_1; whence

$$l = \frac{2\, v^2 \cos \theta \sin \theta}{g}. \tag{29}$$

The range is evidently a maximum when $\theta = 45°$.

54. Orbits of projectiles in a vacuum, whatever be the direction of the initial velocity, are parabolic. Figure 28 shows three such paths, belonging to projectiles from a common point with the same initial velocity. The elevations are 30°, 45°, and 60° respectively.

The range for 60° and 30°, *i.e.* $\overline{Ov}$, is the same, an obvious deduction from equation (29), since $\sin 30° = \cos 60°$, and *vice-versa*.

If an ellipse (Fig. 29) be drawn with the orbit of a vertical projectile for its minor axis and its major axis equal to the range

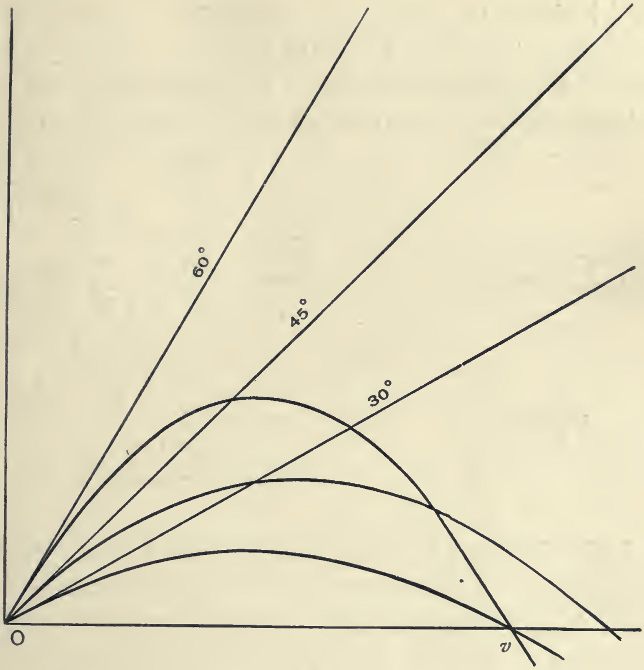


Fig. 28.

of the same projectile fixed at an elevation of 45°, all projectiles from *O* with the same initial velocity, whatever their elevation, will culminate (*i.e.* reach their highest point) upon the periphery of the ellipse.

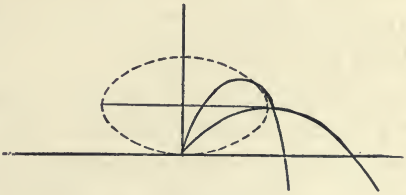


Fig. 29.

It is a fundamental assumption in the analysis of the motions of projectiles that the attraction of the earth is precisely the same upon bodies at rest and bodies in motion, so that the vertical component of the flight of a projectile is the same as that of a body falling from a state of rest. This fact may be readily demonstrated by means of a well-known class-room experiment in which a ball from a spring gun, mounted horizontally (Fig. 30) is so adjusted as to cut the path of a weight falling from a state of repose.

The ball upon leaving the gun severs an electrical connection and releases the weight (*w*), which has been held by an electromagnet in a position in the prolongation of the axis of the gun. Whatever the initial velocity of the ball, it inevitably strikes the falling weight; or if its range be insufficient, the two reach the floor the same instant.

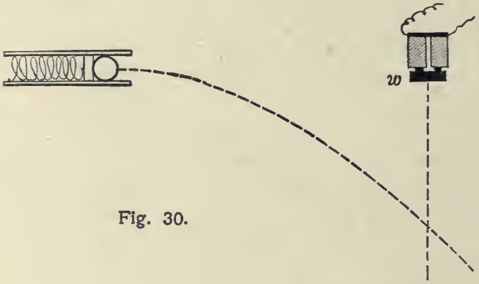


Fig. 30.

55. Projectiles in a resisting medium: the ballistic curve. — Equations (22) to (27) assume a constant acceleration, which condition is not exactly satisfied by a projectile on account of the resistance of the air. The orbits of projectiles, in general, therefore, are not strictly parabolic. The paths which they describe are called *ballistic curves*.

The study of such curves, in the case of projectiles of high velocity, requires the employment of elaborate experimental methods. The paths of bullets and of the other projectiles of warfare, with which the student of gunnery has to deal, vary widely from the parabola.

Projectiles of small velocity, however, have paths which are very nearly parabolic indeed. A convenient case for experimental study is that of a small jet of water or mercury flowing under constant head. By placing the jet in the field of a lantern and projecting an enlarged image upon a screen (Fig. 31) the curve may be drawn and compared with the parabola which would be described in a vacuum. In the case of the curve in the figure, which is from a photograph, the parabolic trace would pass through the series of crosses.

56. Universal gravitation: Newton's law. — Any two particles of matter attract each other with a force F, which is directly proportional to $\frac{mm'}{d^2}$, in which m and m' are the masses of the respective particles, and d is the distance between them. That is,

$$F = k\frac{mm'}{d^2}, \tag{30}$$

in which k is the proportionality factor. The factor k is called the **gravitation constant.**

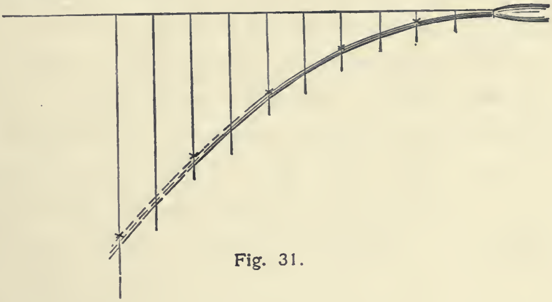


Fig. 31.

Proposition. — The gravitational action of a spherical body on an external particle is the same as if all the matter of the sphere were concentrated at its center. [For proof see chapter on Electrostatics.]

Corollary. — The mutual action of two spheres is the same as if each were concentrated at its center. Therefore, if m and m', equation (30), are the respective masses of two spheres, d the distance between their centers, then F will be their force of attraction. If this force F be observed, then k may be calculated.

57. Determination of the mass of the earth. — The force mg, with which the earth attracts a body of mass m [see equations 16 and 30], is

$$mg = k\frac{mm'}{r^2},$$

whence

$$g = k\frac{m'}{r^2}, \tag{31}$$

in which m' is the mass of the earth, r is the earth's radius, and g is the acceleration of gravity. Thus, equation (31) enables the calculation of m' when k, g, and r are known. This determination was first made by Cavendish, the difficult experimental part of his work being the observation of the extremely small force F, equation (30), of attraction of two lead balls of known mass at a known distance, in order to determine the quantity k. The method employed by Cavendish is briefly described in paragraph 108.

Remark. — The law of gravitation was the result of an effort on the part of Newton to find a common basis for the three laws of planetary motion already discovered by Kepler. Such an effort had been made by others without success. To test the newly formulated law, Newton applied it to the moon with the following result: The value of g at the surface of the earth being $980 \frac{\text{c.m.}}{\text{sec}^2}$, its *value at the moon* stated in modern units is $\frac{4000^2}{240000^2} \cdot 980 \frac{\text{c.m.}}{\text{sec}^2}$ according to Newton's law. Art. 56. This value of g at the moon must be identical to $\dot{v}$ of the moon in its orbit, as determined by equation (32), in which the mean radius in centimeters of the moon's orbit (240000 miles) is written for r, and the reciprocal of the *number of seconds* in a sidereal lunar month is written for n.

CHAPTER IV.

HARMONIC MOTION; STATICS; ENERGY.

58. Uniform motion in a circle. — Consider a particle P (Fig. 32) of mass m, moving n revolutions per second in a circle of radius r. The velocity of P is $v = 2\pi nr$ (a).

From a fixed point O' draw a line $O'P'$ of length l representing v to the scale s, so that $v = sl$ (b). The velocity v is of constant magnitude and at right angles to OP; hence the point P' rotates in a circle n revolutions per second at velocity $2\pi nl$, which velocity represents $\dot{v}$, so that $\dot{v} = 2\pi nsl$ (c). Substituting from (b) and (a) in (c), we have

$$\dot{v} = 4\pi^2 n^2 r. \qquad (32)$$

This acceleration of P is continually directed towards the center of the circular orbit of P, since the velocity of P' is continually parallel to PO.

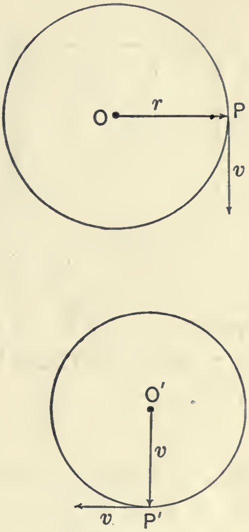


Fig. 32.

Since $v^2 = 4\pi^2 n^2 r^2$, equation (32) may be written:

$$\dot{v} = \frac{v^2}{r}, \qquad (33)$$

or

$$r = \frac{v^2}{\dot{v}}. \qquad (34)$$

Equation (34) expresses the radius of curvature of *any* orbit at a point in terms of the velocity of the particle at the point, and the resolved part of $\dot{v}$ perpendicular to v.

The force F in the direction PO, necessary to keep the particle in its circular orbit, is from equation (15),

$$F=4\pi^2n^2rm. \tag{35}$$

59. Simple harmonic motion is the projection on a fixed straight line of a uniform motion in a circle. Consider a point P (Fig. 33) moving uniformly n revolutions per second in a circle of radius A. Let ω be the number of *radians per second* turned by the line OP (ω is called the angular velocity of OP, see Art. 77); then

$$\omega=2\pi n. \tag{36}$$

Let time be reckoned from the instant that OP coincides with OB, and let OP be the position of the line at time t; then the angle BOP is ωt. If P' be the particle in simple harmonic motion, its distance x from O' is

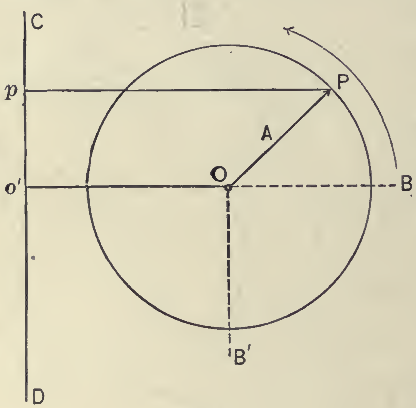


Fig. 33.

$$x=A\sin\omega t, \tag{37}$$

where x is the projection on CD of the rotating line OP.

If time be reckoned from the instant that OP coincides with OB', then the angle $B'OP$ is ωt, and we have

$$x=A\cos\omega t.$$

Position of equilibrium. — The point O' is the position of equilibrium of the vibrating particle P'.

Frequency. — The quantity n is called the frequency of the vibration of P'; it is the number of complete vibrations per second.

Amplitude. — The quantity A, which is the maximum value of x, is called the amplitude of the vibrations.

Phase. —The value of the angle ωt at a given instant is called the *phase* of the simple harmonic motion at that instant.

Period. — The time interval, τ, which elapses during one complete vibration of P', is called the *period* of the vibrations. It is evident that

$$\tau = \frac{1}{n}, \tag{38}$$

or, substituting the value of n from (36), that

$$\tau = \frac{2\pi}{\omega}. \tag{39}$$

Difference in phase. — Consider two particles in simple harmonic motion, of which the respective displacements, x and x', are the projections, on any fixed line, of the lines OP and OP' (Fig. 34) rotating at the same angular velocity ω about the point O, so that the angle θ remains constant. The angle θ is called the *difference of phase* of the two simple harmonic motions. It is evident that $x = A \sin \omega t$ and $x' = A' \sin(\omega t + \theta)$, A and A' being the lengths of OP and OP' respectively, and the time being reckoned from the instant at which the line OP is perpendicular to the fixed line upon which OP and OP' are projected. When the angle θ is zero, the two simple harmonic motions are said to be *in phase;* when $\theta = 90°$ they are said to be in *quadrature*, and when $\theta = 180°$ they are said to be in *opposition.*

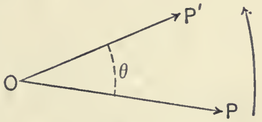


Fig. 34.

60. Harmonic motion and the curve of sines. — If a point P (Fig. 35) have a vertical harmonic amplitude A, and if the paper be moved under P in the direction XO, with uniform velocity, P will describe a curve, the equation of which is

$$y = A \sin \frac{2\pi t}{\tau}. \tag{40}$$

$= A \sin \frac{\omega t}{n}$ $2\pi n t$

$A \sin \frac{2\pi t}{T}$

$= \underline{2\pi t} n$

$= A \sin 2\pi n t$

In this equation, distances along the X axis from intersection to intersection (as $\overline{a, b}$) are taken equal to π, τ is the period, and t has the same significance as in equation (37).

The curve thus produced is *the curve of sines.* It is the characteristic curve of simple harmonic motion.

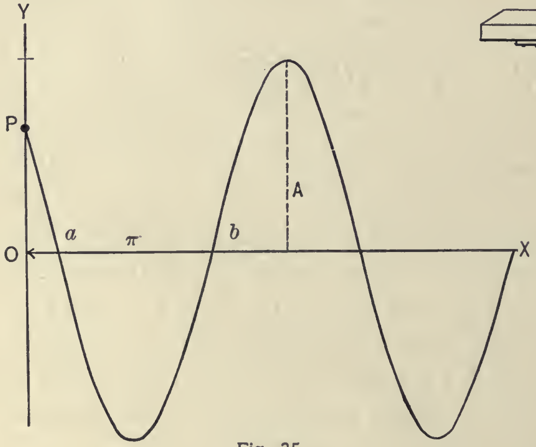


Fig. 35.

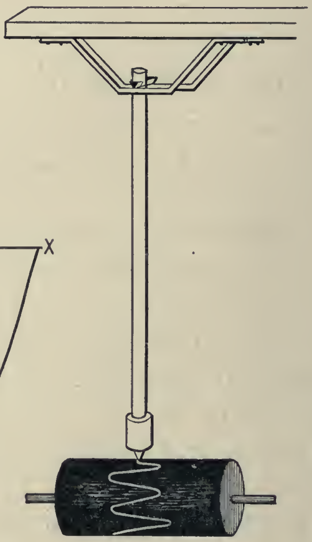


Fig. 36.

Illustration. — If a pendulum be mounted above the cylinder of a chronograph, its oscillations being parallel to the axis of the latter and in the same vertical plane (Fig. 36), it may be made to trace a curve upon the smoked sheet. If the motion of the cylinder be uniform, this curve will be a curve of sines. To obtain such tracings the bob of the pendulum is provided with a movable stylus the point of which will remain in contact with the cylinder throughout the entire oscillation (Fig. 37).

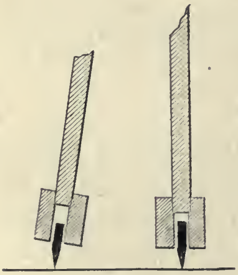


Fig. 37.

It will be shown in the chapters on *Sound* that similar tracings may be obtained with vibrating rods, wires, and other sounding bodies.

61. Velocity and acceleration of a particle in simple harmonic motion. — From equation (37) we find the velocity of the particle to be

$$\frac{dx}{dt} = \omega A \cos \omega t, \tag{41}$$

and from this we find the acceleration of the particle to be

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin \omega t. \tag{42}$$

Equation (41) may be written

$$\frac{dx}{dt} = \omega A \sin\left(\omega t + \frac{\pi}{2}\right). \tag{43}$$

And equation (42) may be written

$$\frac{d^2x}{dt^2} = \omega^2 A \sin(\omega t + \pi). \tag{44}$$

Harmonic variable. — Any quantity which varies as the sine or cosine of a uniformly variable angle is called a harmonic variable. Since the amplitude, period, frequency, and phase of any harmonic variable are defined as in the case of simple harmonic motion, it is evident that the velocity and the acceleration of a particle in simple harmonic motion are harmonic variables. The amplitude of $\frac{dx}{dt}$ is ωA, and it is in quadrature with x; the amplitude of $\frac{d^2x}{dt^2}$ is $\omega^2 A$, and it is in opposition to x.

62. The force necessary to maintain a particle of mass *m* in simple harmonic motion is $F = m\frac{d^2x}{dt^2}$ (see equation 15), whence from (42) we have

$F = ma$

$$F = -m\omega^2 A \sin \omega t. \tag{45}$$

37
39 } — Substituting x for $A \sin \omega t$ and $\frac{2\pi}{\tau}$ for ω, we have, further,

$$F = -\frac{4\pi^2 m}{\tau^2}x. \tag{46}$$

Hence, *if a particle be so conditioned as to be acted on by a force,*

$$F = -ax, \tag{47}$$

in which x is the displacement of the particle from its position of equilibrium and a is a constant, then if the particle be set moving, it will perform simple harmonic motion, such that

$$\frac{4\pi^2 m}{\tau^2} = a, \tag{48}$$

in which m is the mass of the particle, and τ is the period of its motion. Since the amplitude of the motion does not appear in (48), the period τ is independent thereof.

63. The superposition and resolution of simple harmonic motions.

[Superposition of a number of simple harmonic motions of same period and in the same direction, but of different phases.]

Consider a number of simple harmonic motions of the same period and of which the displacements, x, x', and x'', are the projections of the rotating lines OP, OP', and OP'' (Fig. 38).

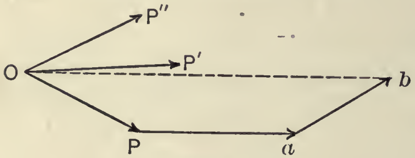


Fig. 38.

The motion, of a particle of which the displacement at each instant is $x+x'+x''$, is said to be a superposition of the three simple harmonic motions. From the figure it is evident that $x+x'+x''$ is at each instant equal to the projection of the line Ob, which is the vector sum of OP, OP', and OP''. Hence the superposition of the three simple harmonic motions, x, x', and x'', is itself simply harmonic, and its amplitude and phase are completely determined by the length and direction of the line Ob.

Corollary. — Any simple harmonic motion may be resolved into a number of component simple harmonic motions of the same period, such that the line, which is the *vector sum* of the lines whose projections represent the displacements of the components, is the line of which the projection represents the displacement of the given simple harmonic motion.

NOTE. — Other cases of the superposition and resolution will be discussed in the chapters devoted to light and sound.

64. The effect of a force. — The effect of a force upon a body depends upon

(*a*) The magnitude of the force.
(*b*) Its direction.
(*c*) Its point of application.

In the study of the action of forces on bodies it is convenient to distinguish two separate actions of a force corresponding to the two possible kinds of motion; namely, *translatory* motion and *rotatory* motion.

65. The tendency of a force to produce translation depends only upon the direction and magnitude of the force, and is the only effective part of a force which acts upon a material particle or upon a body constrained to move without rotation. The combined tendency of a number of forces to produce translation is that of their vector sum.

66. First condition of equilibrium. — In order that a number of forces may have no tendency to produce translatory motion of a body, it is necessary and sufficient that their vector sum be equal to zero (see article 21). This condition requires the sum of the components of the forces in any direction to be equal to zero; so that this first condition of equilibrium may be formulated thus:

$$\begin{aligned} \Sigma X &= 0, \\ \Sigma Y &= 0, \\ \Sigma Z &= 0. \end{aligned} \tag{49}$$

In these equations X, Y, and Z are the components of the various forces in the directions of the axes of reference.

67. The tendency of a force to produce rotatory motion is called **Torque**, sometimes called also **Moment of Force**. The torque action of a force is the only effective part of a force which acts upon a rigid body of which one point is fixed. Let O (Fig. 39) be a fixed point in a rigid body upon which a force F acts at the point p. The torque action T of F about O is

$$T = Fl \sin \theta. \tag{50}$$

It is the vector part of the product Fl. The direction of T as a vector is perpendicular to the plane of F and l; that is, it is in the direction of the axis about which the torque tends to turn the body. This is called the *axis* of the torque. The positive direction of the axis of a torque is considered to be the direction in which a right-handed screw would move if turned by the torque.

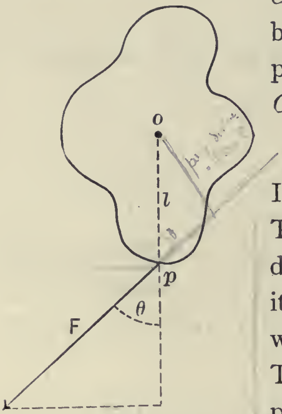


Fig. 39.

68. Components of torque about the axes of reference. — Let T_x, T_y, T_z be the components of the torque action of a force about the x, y, and z axes respectively, and let X, Y, and Z be the components of the force, and x, y, and z the co-ordinates of its point of application. We then have

$$\begin{aligned} T &= Zy - Yz, \\ T_y &= Xz - Zx, \\ T_z &= Yx - Xy. \end{aligned} \tag{51}$$

Proof. — Consider only T_z. From Fig. 40 it is evident that Yx is the torque action of Y about the z axis, and $-Xy$ is the torque action of X about the z axis, it being considered negative because it tends clockwise about the z axis.

69. Representation of a torque by a line. — The line is drawn in the positive direction of the axis of the torque, and of such length as to represent the magnitude of the torque to scale.

70. In equation (50), $l \sin \theta$ is the perpendicular distance from the point O, about which the torque action of the force is reckoned, to the line of action of the force. It follows, therefore, that the torque action of a force is the same wherever its point of application may be in its line of action.

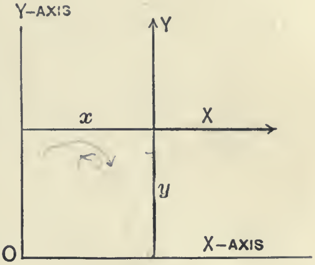
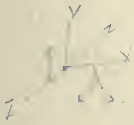


Fig. 40.

71. The total torque action about a given point of a number of forces which act upon a body is equal, both in regard to the direction of its axis and its numerical value, to the vector sum of the torque actions of the separate forces about that point.

72. Second condition of equilibrium. — In order that a number of forces may have no tendency to turn a body upon which they act, it is *necessary* that the vector sum of the torque actions of the separate forces about a point be zero.

This requires the sum of the components of the separate torque actions about each axis of reference to be equal to zero, so that this condition of equilibrium may be formulated thus:



why not $\Sigma(Zy - Yz) + \Sigma(Xz - Zx)$
$+ \Sigma(Yx - Xy) = 0$

$$\begin{aligned} \Sigma(Zy - Yz) &= 0, \\ \Sigma(Xz - Zx) &= 0, \qquad (52) \\ \Sigma(Yx - Xy) &= 0. \end{aligned}$$

73. Couple. — A number of forces may satisfy the first condition and not satisfy the second. In this case the forces tend only to produce rotation, and their action is called a **couple**.

The total torque action of a number of forces which satisfy the first condition of equilibrium, *i.e.* a couple, is the same about all points. For proof consider the z component T_z of this total torque action,

$$T_z = \Sigma(Yx - Xy). \tag{53}$$

Let the origin (about which the torque is reckoned) be transferred to a point of which the x and y co-ordinates are $-a$ and $-b$. Then the x and y co-ordinates of the point of application of each force are increased by a and b respectively, and T_z becomes

$$T_z = \Sigma[Y(x+a) - X(y+b)]; \tag{54}$$

but $$\Sigma Y(x+a) = \Sigma Yx + \Sigma Ya,$$

and $$\Sigma X(y+b) = \Sigma Xy + \Sigma Xb.$$

Since the forces satisfy the first condition, we have

$$\Sigma Ya = a\Sigma Y = 0 \text{ and } \Sigma Xb = b\Sigma X = 0;$$

so that $$\Sigma[Y(x+a) - X(y+b)] = \Sigma(Yx - Xy),$$

and the torque about the new position of the origin is the same as that about its old position. It follows therefore, that the second condition of equilibrium is also *sufficient;* for if the first condition is satisfied and the second condition is satisfied for any one point, it will be for all. The point about which the torque action of a number of forces is reckoned is called the *origin of moments*, and it may be chosen arbitrarily or with a view to convenience in the calculations.

74. Three forces in equilibrium intersect in a point. — *Proof:* Consider the projections of the forces in any plane and choose the origin of moments at the intersection of two of these projections: then the projection of the other force must pass through this point, else the torque action of the forces about an axis perpendicular to the plane cannot be zero; the same being true of any plane, the three forces themselves must intersect in a point.

75. Any number of forces not in equilibrium acting on a body are together equivalent to a single force which is called their resultant; except when the forces constitute a couple. — *Proof:* Given a set of forces in equilibrium, omitting *one* of these forces, the combined action of the others must be equivalent to an equal and opposite force having the same line of action. The exception is also evident, since by omitting one of a set of forces in equilibrium the others cannot constitute a couple. The magnitude and direction of the resultant of a number of forces is determined as their vector sum. The point of application of the resultant is determined by the condition that its torque action about any arbitrary point must be equal to the total torque action of the given forces about that point.

76. The center of mass of a body is the point of application of the resultant of the parallel forces with which gravity acts upon the particles of the body. — *Proof:* Let the origin of co-ordinates O (Fig. 41) be chosen at the center of mass, let the X axis be downwards. The force acting on a particle Δm is $\Delta m \cdot g$, g being the acceleration of gravity, and the torque action of this force about the z axis is $\Delta m \cdot g \cdot y$. The total torque action of the forces on all the particles is $g\Sigma y\Delta m$, which from equation (19) is zero, so that the torque action of the resultant about the center of mass must be zero, and consequently the resultant must pass through that point.

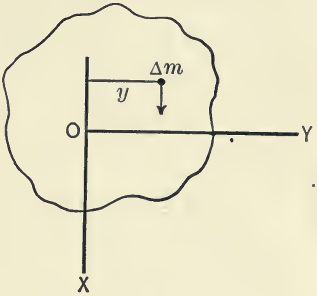


Fig. 41.

77. The **angular velocity** ω of a body is the limiting value of $\frac{\Delta\phi}{\Delta t}$, in which $\Delta\phi$ is the angle turned by the body about its axis of rotation in time Δt. Angular velocity is expressed in *radians per second*. If ω is constant, then $\omega = 2\pi n$, equation (36), in which n is the number of revolutions per second. This is evident from the fact that there are 2π radians in one complete turn. The dimensions of ω are $\frac{1}{t}$. The positive direction of the axis about which a body rotates is considered to be that direction in which a right-handed screw would move if turned with the body. Angular velocity is represented by a line drawn in the positive direction of the axis of rotation and of such length as to represent the magnitude of the angular velocity to scale.

The **angular acceleration** $\dot{\omega}$ of a body is the limiting value of $\frac{\Delta\omega}{\Delta t}$, in which $\Delta\omega$ is the change in ω which takes place during the interval Δt.

78. Motion of a rigid body in a plane. — A rigid body is one which suffers no appreciable distortion under the action of force. A rigid body is said to move in a plane when all points of the body which lie in the plane remain in it. Consider a rigid body *AB* (Fig. 42), moving in the plane of the paper. The position of the body is completely indicated by the position of the line *AB* fixed in the body. This line is called the **index line**.

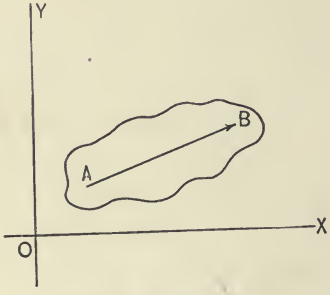


Fig. 42.

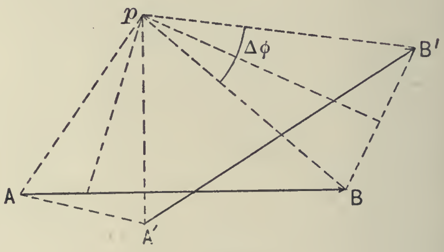


Fig. 43.

79. After any change in the position of a rigid body moving in a plane, a certain line in the body perpendicular to the plane is in its initial position, and the given displacement is equivalent to a rotation about that line as an axis. Let *AB* and *A′B′* (Fig. 43) be the positions of the index line before and after the displacement. Join *AA′* and *BB′*. Erect perpendiculars from the middle points of AA' and BB' intersecting at p. From the similarity of the triangles pAB and $pA'B'$ it is evident that the same part of the body is at p before and after the displacement, and that the line through p perpendicular to the paper is the line about which the body may, by simple rotation, move from its initial to its final position. The angle $\Delta\phi$ of this rotation is the angle subtended by AA' or BB' as seen from p.

80. The instantaneous motion of a rigid body moving in a plane in any manner, is a motion of rotation about a definite line called the **instantaneous axis** of the motion. Let the displacement shown in Fig. 43 be that which takes place in a short interval of time Δt; then $\frac{\Delta\phi}{\Delta t}$ is the instantaneous angular velocity of the body and the line through p, perpendicular to the paper, is the instantaneous axis. During a finite interval of time the motion of a body may be irregular. Forces, however, never being infinite, the motion of a body during an interval approaches uniformity as that interval approaches zero (see equation 15). Therefore the motion of a body during a short interval of time is the simplest motion which can produce the actual displacement which occurs during that interval. In its ultimate consequences this fact is called the **principle of least action.**

81. Composition of angular and linear displacements. — Consider an angular displacement $\Delta\phi$ of a body about the point p (Fig. 44), bringing the point O to O'; and a linear displacement Δl parallel and equal to $O'O$, bringing O' back to O. These two displacements are equivalent to an angular displacement $\Delta\phi$ about O, bringing Op to Op'. Let the distance of p from the line OO' be r; then if $\Delta\phi$ is small, $\Delta l = r\Delta\phi$.

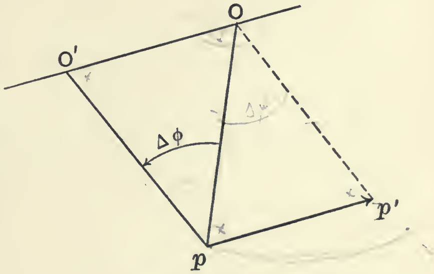


Fig. 44.

82. From Arts. 80 and 81, it follows that the instantaneous motion of a rigid body in a plane may be resolved into a motion of rotation at angular velocity $\frac{\Delta\phi}{\Delta t}$ about an arbitrary point p and a motion of translation at velocity $\frac{\Delta l}{\Delta t}$, such that $\frac{\Delta l}{\Delta t} = r\frac{\Delta\phi}{\Delta t}$. The angular velocity $\frac{\Delta\phi}{\Delta t}$ is equal to the angular velocity of the body at the same instant about the instantaneous axis; and the linear velocity $\frac{\Delta l}{\Delta t}$ is in a direction perpendicular to a line drawn from the arbitrary point to the instantaneous axis.

83. Work and energy. — Whenever the point of application of a force moves in the direction of the force, the force is said to do *work*. Let x be the displacement of a particle in the direction of a force F acting on it; then the work done is

$$W = Fx. \tag{55}$$

If F and x are not parallel, then $F = S \cdot Fx$ (scalar part of product Fx). The dimensions of work or energy are $\frac{ml^2}{t^2}$.

84. Units of energy. — The work done by a force of one dyne working through a distance of one centimeter is called the **erg**. The erg is, for most purposes, an inconveniently small unit, representing, as it does, the amount of work necessary to lift about $\frac{1}{980}$ gram one centimeter against the attraction of the earth. A more suitable quantity would be one comparable in size, for example, to the work done by a force of one kilogram working through a distance of one meter. This quantity, indeed, has been extensively used as a unit by engineers in countries where the metric system is in vogue.

A practical unit, which is about one-tenth of the *kilogram-meter*, and which has the advantages of being independent of locality and precisely commensurate with the erg, is the **joule.** The joule is defined as 10^7 ergs. The *foot-pound* is the work done by a force of one pound working through a distance of one foot.

85. The rate at which work is done in any case is called **power.** From equation (55) we have

$$\frac{dW}{dt}=F\frac{dx}{dt},$$

or $$P = Fv, \tag{56}$$

in which P is the power developed by a force F acting on a body moving at velocity v in the direction of the force. If F and v are not parallel, then $P = S \cdot Fv$. The dimensions of power are $\frac{ml^2}{t^3}$.

86. Units of power. — The c. g. s. unit of a power is *one erg per second.* The practical unit corresponding to the joule is the *watt.* It is *one joule per second.* The *horse power* is 550 foot-pounds per second.

The work done by a torque T upon a body, when it is displaced through an angle ϕ about the axis of the torque, is

$$W = T\phi. \tag{57}$$

Proof. — Consider a torque, due to a force F (Fig. 45), acting continually at right angles to an arm of length l; then $T = Fl$. When the body is turned through an angle ϕ, the point of application of the force will move a distance ϕl in its own direction, and from equation (55) $W = F\phi l$, or $W = T\phi$. Q.E.D.

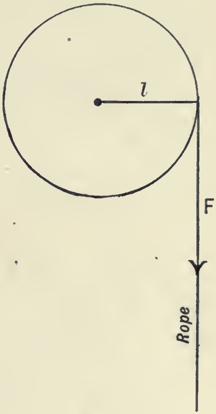


Fig. 45.

87. The power developed by a torque acting upon a body rotating about the axis of the torque is found by differentiating equation (57), whence

$$\frac{dW}{dt}=T\frac{d\phi}{dt},$$

or $$P = T\omega. \tag{58}$$

88. Measurement of power. — Power may be measured in the following ways:

(*a*) By measuring a force and the velocity of the body upon which it acts. Equation (56) is used to calculate P.

(*b*) By measuring a torque and the angular velocity of the body upon which it acts. Equation (58) is used to calculate P.

(*c*) By measuring an electromotive force and the electric current which it maintains.

89. The energy of a system of particles. — When a system of particles by virtue of its **state of motion** is capable of doing work, it is said to possess or store energy. Its energy is measured by the work which it can do.

90. The energy stored in a system by virtue of its **velocity configuration** is called its **kinetic** energy. For example, a moving train stores energy because of its motion; a magnetic field stores energy because of the turbulent motion of the ether which constitutes the magnetic field.

91. The kinetic energy of a particle is

$$W = \tfrac{1}{2} mv^2. \tag{59}$$

Proof. — The kinetic energy of a particle is measured not only by the work which it can do when stopped, but also by the work required to establish its motion. Let a constant force F act upon a particle of mass m, starting it from rest. This force will do work on the particle at a rate $\frac{dW}{dt} = mv\frac{dv}{dt}$ (see equations (15) and (56)); whence $dW = mvdv$, and $W = \frac{1}{2} mv^2$.

The kinetic energy of a system of particles is

$$W = \tfrac{1}{2} \Sigma mv^2, \tag{60}$$

the summation being extended to all the particles.

92. The energy stored in a system by virtue of its position configuration is called the **potential energy** of the system. For example, a weight stores energy by virtue of its position relative to the earth; a bent spring contains stored energy by virtue of its configuration; an electrical field contains stored energy by virtue of a distortion or altered configuration of the ether which constitutes the electrical field.

It is necessary to adopt arbitrarily some certain configuration of a system, called the *zero configuration*, for which the potential energy, W_0, is an unknown constant, and to reckon the potential energy of any other configuration as W_0 *plus* the work necessary to change the system from the zero configuration to that configuration.

93. Mutual relation between kinetic and potential energy of a closed system. — Any change in the kinetic energy of a closed system is accompanied by an equal and opposite change in the potential energy of the system. This is evident when we consider that any increase in the kinetic energy of a closed system must be equal to the work done by the mutual force action of the particles of the system while those particles are being displaced; and that the displacements of the particles bring the system into a new configuration such that an equal amount of work must be done against the mutual forces to restore the system to its initial configuration.

94. The principle of the conservation of energy. — The work required to change a system from its zero configuration to any other configuration is completely independent of the intermediate stages through which the system passes. For example, the energy required to change a gram of water from its freezing-point at atmospheric pressure to its boiling-point at atmospheric pressure is the same whether the water be simply and ordinarily heated or subjected to any number of chemical and physical changes in any manner whatever during the process of bringing it to its boiling-point, work done on the water being considered positive, and work extracted from the water being considered negative.

95. The statement in Article 94 is strictly a part of the definition of potential energy, since without that statement the whole of Article 92 is invalid. Keeping this in mind, we see from Article 93 that the principle of the conservation of energy may be stated thus: *The total energy of a closed system is constant.*

A change in the total energy of any system must be due to an exchange of energy between that system and another. This is evident at once if we consider the two systems together as a closed system.

96. By way of illustration let us consider one of the simplest cases of the transformation of energy from potential to kinetic form and *vice versa;* that, namely, which occurs during the oscillations of a pendulum. Let the latter consist of a metallic bob of mass m, suspended from a point O (Fig. 46), around which it is free to revolve.

When the pendulum has its center of mass at the lowest point, and coincides with the vertical line Ob_0, we may consider that it has its *zero configuration of position* (Art. 92). Let the stored potential energy due to that position be W_0, and let the bob be raised to b_1 by the expenditure of an amount of work

$$W = Mgx. \tag{61}$$

This work is entirely independent of the path along which M has been carried in order to transfer it from b_0 to b_1. The potential energy in the new position is $W_0 + W$.

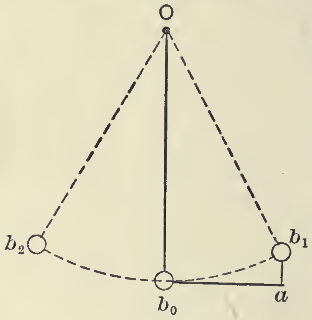


Fig. 46.

Let the pendulum be now released and allowed to swing. We may for the moment overlook the resistance of the atmosphere, etc., and regard the pendulum as a closed system, the potential energy of which varies at every quarter oscillation between W_0 and $W_0 + W$.

The kinetic energy of the system also varies; from zero, when the pendulum reaches its greatest elongation at Ob_1 or Ob_2 and comes to rest in what may be called the *zero velocity configuration*, to a maximum at Ob_0.

To show for further illustration that the decrement of potential energy during a short interval of time is equal to the simultaneous increment of kinetic energy, the pendulum and the attracting earth being considered a closed system, consider any short element, ΔS (Fig. 47), of the path of the bob. Let v be the velocity of the bob at the instant at which it enters upon this element, Δt the time interval required for it to pass over the element, and Δv the gain in velocity during the interval Δt. Then $\Delta S = v \cdot \Delta t$, and $\Delta x = \Delta S \cdot \sin\theta = v \,\mathrm{Sin}\, \theta \cdot \Delta t$. Multiplying Δx by the weight, mg, of the bob, we have

$$\Delta W = mgv \,\mathrm{Sin}\, \theta \cdot \Delta t \qquad (62)$$

as the decrement of potential energy during Δt. The acceleration $\dot{v}$, of the bob in its path is the component, $g \sin\theta$, of g in that direction so that $\Delta v = \dot{v} \cdot \Delta t = g \sin\theta \cdot \Delta t$. The kinetic energy of the bob at the beginning of the interval is $\frac{1}{2} mv^2$, and at the end of the interval is $\frac{1}{2} m(v + \Delta v)^2$. Therefore, putting $g \,\mathrm{Sin}\, \theta \cdot \Delta t$ for Δv we have

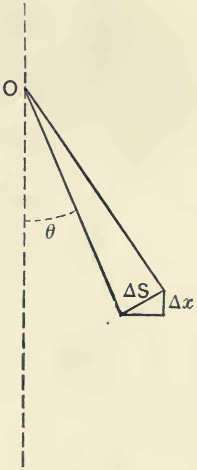


Fig. 47.

$$\Delta W = mgv \sin\theta \cdot \Delta t, \qquad (63)$$

as the increment of kinetic energy during Δt.

Since the changes of kinetic and potential energy during every short interval compensate for each other, this compensation must hold for long intervals as well. Therefore the potential energy, mgx, of the bob at b (Fig. 46), must be equal to the kinetic energy, $\frac{1}{2} mv^2$, of the bob at b_0, that is, $\frac{1}{2} mv^2 = mgx$, or $v = \sqrt{2gx}$, which is the velocity the bob would gain in falling freely from b_1 to a.

In point of fact, a swinging pendulum is not, strictly speaking, a closed system. At every oscillation it imparts energy; by atmospheric resistance, if it swings in the air; by electromagnetic induction, if it happens to swing in a magnetic field and by the frictions of its bearings, to other portions of the greater system to which it belongs. Finally when it comes to rest in its zero configuration of position, its total energy is reduced to the initial value, W_0.

Further discussion of the pendulum must be deferred until the principle of the moment of inertia has been considered. It will be found in Chapter V.

CHAPTER V.

MOMENT OF INERTIA; THE PENDULUM.

97. Moment of inertia. — The kinetic energy W of a rigid body rotating about a fixed axis is proportional to the square of its angular velocity, ω. For the linear velocity of each particle of the body being proportional to ω, the kinetic energy of each particle, and, therefore, the total kinetic energy, is proportional to ω^2. We may, therefore, write

$$W = \tfrac{1}{2} K\omega^2, \tag{64}$$

in which $\frac{1}{2}K$ is the proportionality factor. The quantity K is called the *moment of inertia* of the body about the given axis.

Dependence of moment of inertia upon mass and dimensions of a body. — Consider a rigid body (Fig. 48) rotating about the axis p, perpendicular to the paper, at an angular velocity ω. The linear velocity v of a particle Δm, distant r from p, is $v = r\omega$, and the kinetic energy of this particle is $\frac{1}{2}\Delta m r^2\omega^2$, so that the total kinetic energy of the body is $W = \frac{1}{2}\omega^2\Sigma r^2\Delta m$. Comparing this with (64), we find

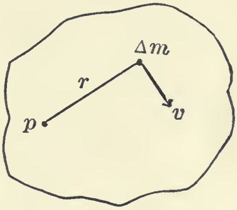


Fig. 48.

$$K = \Sigma r^2 \Delta m. \tag{65}$$

98. Radius of gyration. — Let $\frac{m}{n}$ be the mass of each particle of the body, m being its total mass and n a large number. Writing $\frac{m}{n}$ for Δm in (65), we have $K = m\frac{1}{n}\Sigma r^2$. Putting

$$r_1^2 = \frac{1}{n}\Sigma r^2, \tag{66}$$

we have

$$K = mr_1^2. \tag{67}$$

F

The length r_1 is called the **radius of gyration** of the body; it is equal by (66) to the square root of the average square of the distances of the various particles (of equal mass) of the body from the axis.

99. Relation between moments of inertia about parallel axes. — Let K be the moment of inertia of a body of mass m about a given axis passing through the center of mass of the body, and K' its moment of inertia about another axis parallel thereto and distant a from the center of mass; then

$$K' = K + a^2 m. \tag{68}$$

Proof. — Let C (Fig. 49) be the center of mass of the body chosen as the origin of co-ordinates, and let p be the axis, perpendicular to the paper, about which K' is reckoned. Consider a particle Δm, distant r from C, distant r' from p, and of which the co-ordinates are x y. We have $r'^2 = r^2 + a^2 - 2\,ra \cos\phi$, and $K' = \Sigma r'^2 \Delta m$ (65); whence

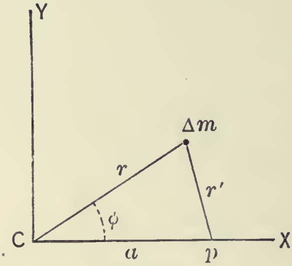


Fig. 49.

$$K' = \Sigma r^2 \Delta m + a^2 \Sigma \Delta m - 2\,a \Sigma r \cos\phi \Delta m.$$

From (65) we have $K = \Sigma r^2 \Delta m$; also $a^2 \Sigma \Delta m = a^2 m$, and $\Sigma r \cos\phi \Delta m = \Sigma x \Delta m = 0$ from equation (19).

The following are the moments of inertia of some regular solids: —

Body.	Value of K.
Sphere (axis a diameter)	$\frac{2}{5}\,mr^2$
Cylinder (axis is the axis of figure)	$m\dfrac{r^2}{2}$
Ring (axis the axis of figure, *i.e.* through center and perpendicular to plane of ring; [r_e and r_i are outer and inner radii])	$m\dfrac{r_e^2 + r_i^2}{2}$
Rod (axis at end and perpendicular to the axis of figure; [l = length of rod])	$m \cdot \dfrac{l}{3}$
Cone (axis the axis of figure; [r is the radius at the base]),	$\frac{3}{10}\,mr^2$

100. Relation between torque and angular acceleration. — In equation (64), K being constant, if ω is changing, we may write $\frac{dW}{dt} = K\frac{d\omega}{dt} \cdot \omega$. Comparing this with equation (58), and remembering that $\frac{dW}{dt}$ is here the rate at which work is done on the body to cause ω to change, we have $T = K\frac{d\omega}{dt}$, or

$$T = K\dot{\omega}. \tag{69}$$

The product $K\omega$ is called the **angular momentum** of the body. Equation (69) shows that the torque which acts upon a body is equal to the rate at which its angular momentum changes; also that the angular momentum of a body is constant when no torque acts upon it.

101. Simple harmonic rotary motion. — When a body moves in such a way that its angular displacement ϕ (about a fixed axis), from its position of equilibrium, is proportional to the sine or cosine of a uniformly variable angle, the body is said to perform simple harmonic motion. All equations and propositions for simple harmonic linear motion hold in the present case, provided that we write angular displacement ϕ for linear displacement x, angular velocity and acceleration for linear velocity and acceleration, and moment of inertia for mass. Hence we have the following proposition from Art. 62:

Proposition. — Let a body be so conditioned that when turned through an angle ϕ from its position of equilibrium, it is urged back by a torque,

$$T = -b\phi, \tag{70}$$

in which b is a constant; then, if the body be started, it will vibrate in such a way that

$$\frac{4\pi^2 K}{\tau^2} = b, \tag{71}$$

in which K is the moment of inertia of the body about the axis of its motion, and τ is the period of its vibrations.

102. Kinematics of a rigid body with one point fixed.

Proposition. — If a rigid body having a fixed point be displaced in any manner whatever, a certain line in the body will be in its initial position, and the given displacement will be equivalent to a rotation about this line as an axis.

Proof. — Consider a spherical shell of the body having its center at the fixed point. If *AB* (Fig. 50) be an arc of a great circle on this sphere, the position of *AB* fixes the position of the body, and *AB* is called the *index line.* Let the movement of the body bring *AB* to *A′B′*. Connect *AA′* and *BB′* by arcs of great circles. Draw great circles bisecting *AA′* and *BB′* at right angles. The point *p* so determined has the same position relative to *AB* as to *A′B′*. A line drawn from the center of the sphere to *p* is thus the axis about which the given movement can be produced by rotation.

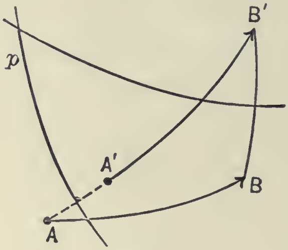


Fig. 50.

Corollary. — The instantaneous motion of a rigid body about a fixed point is a motion of simple rotation at a definite angular velocity about a definite line, called the **instantaneous axis** of the motion.

103. Kinematics of a rigid body entirely free.

Proposition. — Any displacement of a rigid body is equivalent to a translation along a certain line, combined with a rotation about that line as an axis. This is called a **screw displacement.**

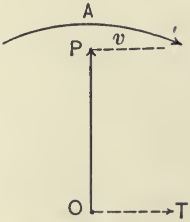


Fig. 51.

104. The gyroscope. — Consider a body rotating rapidly at a constant angular velocity ω about its axis of symmetry *OP* (Fig. 51), this axis at the same time turning slowly about *O* in the direction of the arrow *A*. Let the line *OP* represent the angular momentum $K\omega$ of the rotating body to scale. Then the velocity of P will represent the rate at which the angular momentum of the body changes, viz. the torque acting upon the body. Hence the body will move in the prescribed manner if set moving about its axis of symmetry, and then acted upon by a torque, the axis OT of which is continually at right angles to OP.

This action of the gyroscope is explained directly by considering the motion of the particles of the rotating body.

Consider a circular disk (Fig. 52) rotating rapidly in the direction *ef*, at an angular velocity ω, about an axis O perpendicular to the plane of the disk. Suppose this *axis* to turn slowly at an angular velocity ω' about the fixed line AB, in such direction as to bring the limb C of the disk towards the observer. Consider the velocity v of the successive particles as they pass the point P. If the axis O of the disk were stationary, this velocity v would be constant both in magnitude and direction; but since the axis O turns about AB, v changes. If we draw a line $O'P'$ representing v to scale, this line $O'P'$ will sweep round a circular cone at regular velocity ω'. The velocity of the point P' in the diagram is towards the observer, and represents $\dot{v}$ to the same scale that $O'P'$ represents v; hence $\dot{v} = \omega' v \cos\theta$. Substituting ωr for v, and multiplying by the mass of the particle at P, Δm, we have $\Delta F = \omega' \omega r \cos\theta \Delta m$, which is the force, aside from such force as keeps the particle in its circular path, necessarily acting on the particles as they pass through P. These

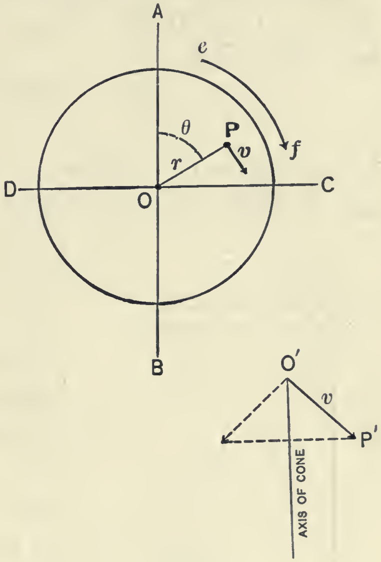


Fig. 52.

forces ΔF are all towards the observer above DC, and away from the observer below DC, and therefore their resultant is a torque about the axis DC.

105. The torsion pendulum. — A body hung by a wire is in equilibrium when the wire is untwisted. When turned through an angle ϕ from its position of equilibrium, about the wire as an axis, the body will be urged back by a torque T which is proportional to ϕ, so that we have

$$T = -b\phi. \qquad (70\ bis)$$

The quantity b is called the **constant of torsion** of the wire. See Art. 156. If the body be started, it will vibrate in such a way that

$$\frac{4\pi^2 K}{\tau^2} = b, \qquad (71\ bis)$$

in which K is the moment of inertia of the body, and τ is the period of its vibrations. The balance wheel of a watch is a torsion pendulum, the governing torque being due to the hair spring.

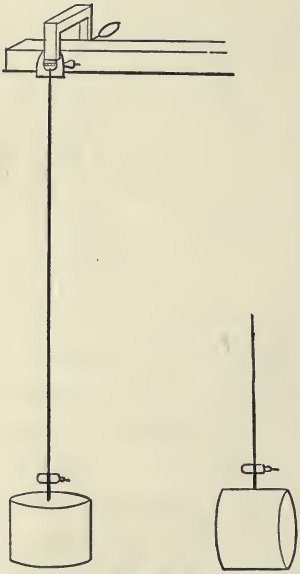


Fig. 53.

106. The torsion pendulum in the experimental study of moments of inertia. — In case b is known, by previous experiments with the same wire and a suspended weight the moment of inertia of which is known, we may, after determining τ, compute K from equation (71).

By this method the various statements and principles given in Arts. 97 to 99 may be empirically verified.

Example. — If a cylindrical weight (Fig. 53) is suspended from a vertical wire, the axis of its motion being the axis of the cylinder, and if the nature of the suspension be such that this motion can take place only by virtue of the twisting of the wire, the arrangement will constitute *a torsion pendulum.*

Suppose it be required to find the ratio of the length of a homogeneous cylinder to its radius, such that the period of oscillation will be the same whether the axis of gyration be the axis of figure or perpendicular to the same and passing through the center of the mass.

In the former case

$$K_1 = \frac{m}{2} \cdot R^2, \tag{72}$$

where R is the radius of the cylinder and the radius of gyration is

$$r_1 = \frac{R}{\sqrt{2}}.$$

In the latter case

$$K_2 = \frac{m}{12}(L^2 + 3R^2), \tag{73}$$

where L is the length of the cylinder.

From equation (71) we have as the condition of equality of period, $K_1 = K_2$,

or
$$\frac{m}{2}R^2 = \frac{m}{12}(L^2 + 3R^2).$$

$$\therefore\ L = R\sqrt{3}.$$

The radius of gyration in the latter case is

$$\sqrt{\frac{L^2}{12} + \tfrac{1}{4}R^2} = \frac{R}{\sqrt{2}},$$

a value identical with the above.

If a weight of the form described be suspended in the two positions, the identity of values of the moments may readily be verified.

107. The torsion pendulum in the determination of the constant of torsion. — In this case b is the unknown quantity. A weight of known moment of inertia is suspended by means of the wire the torsional elasticity of which is to be tested, and the period is observed. If the length and radius of the wire are also measured, the torsional coefficients for the material of which the wire is composed may be computed as well as the value of the constant of torsion (b) for the individual wire. (See further, paragraphs 152 *et seq.*)

108. The torsion pendulum in the measurement of small forces. — When used for this purpose, the torsion pendulum is called a *torsion balance.* With such an instrument, the essential features of which are depicted in Fig. 54, Coulomb (1785–1788) investigated the laws of electrostatic attraction and repulsion and the mutual actions of magnets.

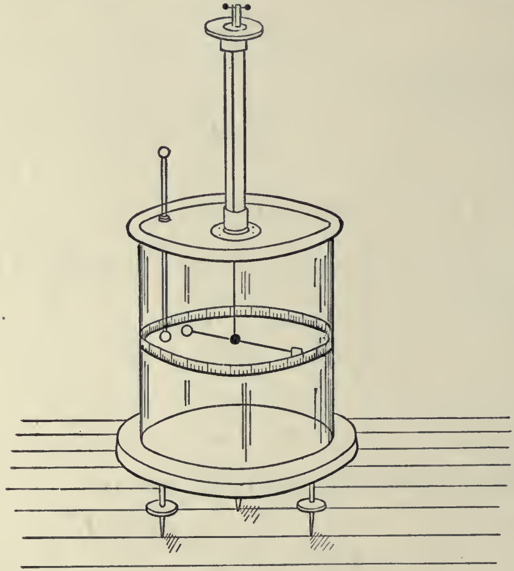


Fig. 54.

The method of using the torsion balance consists in twisting the suspension wire at the upper end until the torque exerted upon the attracted or repelled body m_1 is balanced. The angle through which the wire is twisted is measured. This is proportional to the force thus exerted (Eq. 70). The proportionality factor b is determined by suspending from the wire a weight of known moment of inertia and observing the period of oscillation.

Cavendish's experiments. — The most notable instance of the use of the torsion balance was in the celebrated experiments of Cavendish[*] (1798), upon the density of the earth. (See Art. 57.)

Cavendish's apparatus consisted of a torsion balance (Fig. 55) in which two leaden balls at the end of equal lever arms were suspended by a long wire. The mass of each of these in the original experiment was 730 grams. When two larger spheres of lead, each of which weighed 158,000 grams, were brought near, the gravitational attraction was sufficient to produce a measurable deflection of the suspended balls.

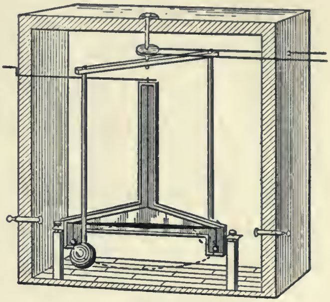


Fig. 55.

The masses and the distances between their centers of mass being known, also the distance to the center of the earth, a comparison of the force indicated by the torsion balance with that exerted by the earth upon the suspended balls, sufficed for the computation of the mass of the latter, and indirectly of its mean density.

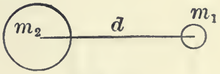


Fig. 56.

If m_1 and m_2 (Fig. 56) be the masses of the leaden spheres brought into proximity, and d be the distance between them, we have

$$F = k\frac{m_1 m_2}{d^2}, \tag{74}$$

where k is the gravitational constant; compare Eq. 30.

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[*] Repeated by Bailly (1842), Reich (1852), Cornu and Baille (1878)

The earth's force upon m_1 is, however (see Art. 57):

$$m_1 g = k\frac{Mm_1}{r^2}, \tag{75}$$

and

$$\frac{F}{m_1 g} = \frac{m_2 r^2}{Md^2},$$

where M is the mass of the earth, and r is its radius.

Boys' experiment. — The very remarkable elastic properties of filaments drawn from fused quartz, a discovery which we owe to the English physicist Boys, has made it possible to repeat the Cavendish experiment with very much smaller masses.

By means of a torsion balance with a long and very fine quartz fiber, Boys was able to show distinctly the gravitational forces between two masses of metal of the size of ordinary bullets.

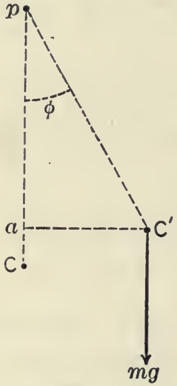

Fig. 57.

109. The gravity pendulum. — Consider a rigid body of mass m suspended so as to be free to turn about a horizontal axis p. Let C (Fig. 57) be the center of mass of the body, vertically below p, when the body is in equilibrium. Let the body be turned to one side through the angle ϕ. The torque T tending to bring the body back into its vertical position is

$$T = -mgx \sin\phi, \tag{76}$$

in which x is the distance pC. When ϕ is small, then $\sin\phi = \phi$, and equation (76) becomes

$$T = -mgx\phi. \tag{77}$$

Comparing this with equations (70) and (71), we find that

$$\frac{4\pi^2 K}{\tau^2} = mgx, \tag{78}$$

in which K is the moment of inertia of the body about the axis p, τ is the period of its vibrations, and g is the acceleration of gravity. A rigid body suspended in this manner is called a **gravity pendulum** or simply a **pendulum.**

110. The equivalent length of a pendulum. — Imagine the pendulum just described to consist of a small bob of mass m suspended by a thin rod of length l and of which the mass is negligible. Then $K = ml^2$ from (65), and equation (78), when solved for l, becomes

$$l = \frac{\tau^2 g}{4\pi^2}. \tag{79}$$

Such a pendulum is called a **simple pendulum.** The quantity $\frac{\tau^2 g}{4\pi^2}$ is for *any* pendulum called its **equivalent length.** Solving equation (78) for $\frac{\tau^2 g}{4\pi^2}$, we have

$$l = \frac{K}{mx}, \tag{80}$$

in which l is the equivalent length of the pendulum. The point in the line pC (Fig. 57) at a distance of l from p is called the **center of oscillation** of the pendulum. This point is also called the **center of percussion,** for the reason that if the pendulum is started or stopped by a force applied at this point, no side force is produced on the axis.

111. The reversion pendulum. *Determination of gravity.* — Consider a body of mass m, its center of mass at O, Fig. 58. Let O', O and O'' be co-linear points, let τ' and τ'' be the vibration periods of the body swung as a pendulum from O' and O'' respectively, and let K, K', and K'' be the moments of inertia of the body about O, O', and O'' respectively. From equation (78) we have

$$\frac{4\pi^2 K'}{\tau'^2} = mgx \tag{i}$$

and

$$\frac{4\pi^2 K''}{\tau''^2} = mgy. \tag{ii}$$

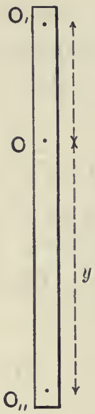

Fig. 58.

From equation (68) we have

$$K' = K + x^2m, \tag{iii}$$

$$K'' = K + y^2m. \tag{iv}$$

Substituting these values of K' and K'' in (i) and (ii), we have

$$\frac{4\,\pi^2(K + x^2m)}{\tau'^2} = mgx \tag{v}$$

and

$$\frac{4\,\pi^2(K + y^2m)}{\tau''^2} = mgy. \tag{vi}$$

Eliminating $\frac{K}{m}$ from (v) and (vi), we have

$$\frac{4\,\pi^2\,(x^2 - y^2)}{x\tau'^2 - y\tau''^2} = g. \tag{81}$$

If $\tau' = \tau''$, we may cancel $(x - y)$, *if* $(x - y)$ *is not equal to zero*, giving

$$\frac{4\,\pi^2(x + y)}{\tau'^2} = g. \tag{82}$$

(1) If the pendulum has been adjusted, by repeated trial, so that $\tau' = \tau''$, then equation (82) enables the calculation of g when $(x + y)$ and τ' have been observed.

(2) If the pendulum has not been adjusted, equation (81) enables the calculation of g when x, y, τ', and τ'' have been observed.

(3) If the pendulum has been roughly adjusted so that τ' and τ'' are nearly equal, then *equal and opposite errors in x and y very nearly annul each other in their influence upon the value of g as calculated by equation* (81). Therefore, equation (81) gives g very accurately when τ' and τ'' are nearly equal, $(x + y)$ being measured with great accuracy and x measured roughly; the value of y being taken from $(x + y) - x$ so that its error may counteract the error due to the roughly determined value of x. The position of the center of mass, *O*, is found with sufficient precision for this rough measurement of x, by balancing the pendulum horizontally on a knife edge.

This method for determining gravity was first used by Kater, see Art. 115.

NOTE. — When $x=y$ equation (81) becomes indeterminate $\frac{0}{0}=g$, and in this case equation (82) is not necessarily true, since it has been derived from (81) by cancelling $(x-y)$, which is zero.

112. Conjugate relation between point of support and center of oscillation of a pendulum. — Solving equation (82) for $x+y$ and putting τ for the common value of τ' and τ'' we have $x+y=\frac{T^2g}{4\pi^2}$ which being identical to $l=\frac{t^2g}{4\pi^2}$ (79), shows that O'' is the center of oscillation of the pendulum when O' is the point of support and *vice versa*.

113. Identity of center of oscillation and center of percussion of a pendulum. — Consider a pendulum suspended at O′, Fig. 59, with its center of mass at O distant x below O′, and its center of percussion at O″ distant l below O′. If the pendulum turns about O′ at angular velocity ω, then $v=r\omega$ is the linear velocity of a particle Δm of the pendulum distant r below O′, and $\dot{v}=r\dot{\omega}$ is the acceleration of this particle, so that $F=\Delta m\cdot\dot{v}=r\dot{\omega}\cdot\Delta m$ is the side force necessarily acting upon the particle Δm. Similar forces must act on all the particles of the pendulum. The resultant of all such forces must pass through the center of percussion, for that is, by definition, the point at which a *single* force acting upon the pendulum will start it moving; therefore the total torque action of these forces about O″ must be zero. That is, $\Sigma(l-r)r\dot{\omega}\cdot\Delta m=0$, or $l\Sigma r\cdot\Delta m=\Sigma r^2\Delta m$. But $\Sigma r^2\Delta m=K$ is the moment of inertia of the pendulum about O′ and $\Sigma r\cdot\Delta m=mx$ from equation (18), m being the total mass

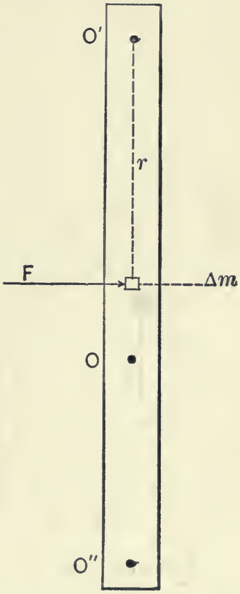

Fig. 59.

of the pendulum. Therefore $lmx = K$, or $l = \frac{K}{mx}$ which being identical to equation (80) shows that o'' is also the center of oscillations of the pendulum.

114. Experimental verification of the law of the pendulum. — Equation (79) embodies what are often called the laws of the pendulum: viz. the relations, $\tau^2 : l$, and $\tau^2 : \frac{1}{g}$. From the absence of any factor depending upon the amplitude, the additional *law of isochronism*, said to have been discovered by Galileo by means of observations upon a swinging lamp in the cathedral of Pisa (1583), may be inferred. Galileo's law of equal times is an approximation. It is in accordance with equation (79), but that equation is based upon the assumption, true only for an infinitesimal arc, that $\sin \phi = \phi$. Where the amplitude is small but measurable, a closer approximation to the true value is obtained by the introduction of a correction factor; equation (79) solved for t taking the form *

$$\tau = 2\pi\sqrt{\frac{l}{g}}\cdot \left(1 + \frac{\phi^2}{16}\right). \tag{83}$$

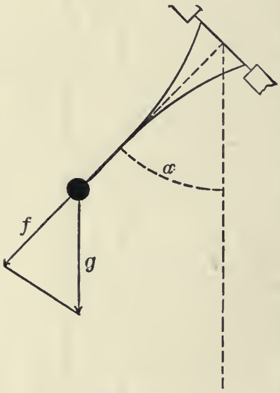

Fig. 60.

The approximate validity of the law of equal times is readily verified by observations upon a pendulum of any form; while the relation $\tau^2 : l$ may be verified by comparing the periods of two simple pendulums, one of which is adjustable as to length.

To test the relation $\tau^2 : g$, an apparatus devised by Mach is frequently used. It consists of a pendulum (Fig. 60), the axis of support of which can be inclined at an angle with the horizon. When the line joining the axis of sup-

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* The full form is

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left[1 + (\tfrac{1}{2})^2 \sin^2 \frac{\phi}{2} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \sin^4 \frac{\phi}{2} + \left(\frac{1 \cdot 3 \cdots 2n - 1}{2 \cdot 4 \cdots 2n}\right)^2 \sin^{2n} \frac{\phi}{2} \cdots\right].$$

port and the center of oscillation makes an angle α with the vertical, the force under which the pendulum acts is

$$f = g \cos \alpha,$$

and the equation of the pendulum becomes

$$\tau = 2\pi\sqrt{\frac{l}{g \cos \alpha}}.$$

115. Determination of the acceleration of gravity by means of the pendulum.

Borda's Method. — The period τ of a pendulum of simple geometrical form, such that its moment of inertia may be computed, is observed, and g is calculated by means of equation (78).

The apparatus employed by Borda, who made the earliest precise determinations of gravity (at Paris, 1792), consisted of a pendulum with a spherical bob mounted directly in front of the pendulum of a clock the rate of which was known. By observation of the times when the two pendulums, which were nearly of the same length, were in coincidence, the ratio of the periods was determined with great accuracy.

The total length of the pendulum was determined by bringing up an adjustable platform from below until tangent with the bob, and measuring its distance from the axis of support. This distance, less the radius of the bob, gave the length (x) to center of mass.

The moment of inertia K of a sphere, about an axis at a distance x from the center of mass, is

$$K = M\left(\tfrac{2}{5}R^2 + x^2\right),$$

where R is the radius of the bob. Knowing thus the value of $\frac{K}{M}$ and of x and τ, g may be calculated from equation (78).

The method of determining the period of a pendulum, referred to above, is called the *method of coincidences*. Where the pendulums to be compared have nearly the same rate and the observations can be continued over several hours, it is one of the most exact of all processes of physical measurement. It is, for example, a simple matter to obtain the *relative* period to within the millionth of a second.

Fig. 61.

Kater's Method.—The principle of the reversibility of the centers of support and oscillation, discovered by the Dutch mathematician Huyghens, in the seventeenth century, was first utilized in the determination of gravity by Kater (1818). See Arts. 111 and 112.

The reversible pendulum (Fig. 61) has two sets of adjustable knife edges. If, when it is suspended from one of these (*aa*), the other (*bb*) passes through the center of oscillation, the pendulum may be reversed and suspended from *bb*, without altering the period. The distance from *aa* to *bb* is then the equivalent length. The distribution of mass is adjusted by successive trials until the above condition is accurately fulfilled.

The advantage of the method lies in the fact that l is a quantity capable of direct measurement.

After adjustment the period is finally determined by the method of coincidences, or by automatic record upon a chronograph.

In its best form, the reversible pendulum is so designed as to be symmetrical in shape, with the knife edges disposed at nearly equal distances from the center of volume. The influence of the surrounding air will then be the same, whether the pendulum is suspended from *aa* or from *bb*. The form, moreover, should be that of some simple geometrical figure for which the influence of the atmosphere (a very small but an appreciable effect) can be computed. In the most refined determinations the pendulum swings in an air-tight chamber, the atmosphere within which is maintained at a constant pressure.

The above considerations have led to the abandonment of the earlier form of pendulum (Fig. 61) in favor of that shown in Fig. 62. The latter is a hollow cylinder with rounded ends. One end is loaded so that the center of mass does not correspond with the center of volume.

For the purposes of the relative measurement of gravitation, *i.e.* for the comparison of this quantity in various localities with its value at some station where *g* is accurately known, small non-reversible pendulums of the form shown in Figs. 63 and 64 are used.* Their period is $\frac{1}{2}$ second, or even $\frac{1}{4}$ second. The apparatus for the determination of *g* is thus reduced to an easily portable size.

Fig. 62.

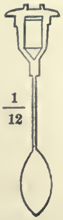

Fig. 63.

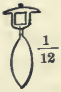

Fig. 64.

Value of g in various parts of the surface of the earth.— The value of the acceleration due to gravity in a number of localities being given, it is possible to construct a formula, based upon our knowledge of the dimensions of the earth and of its motion, by means of which *g* may be computed for other places.

The following equation gives a result as accurate as can be expected from any expression which does not take cognizance of local irregularities and of the lack of homogeneity in the crust of the earth:

$$g = 978.07 + 5.08 \sin^2 \lambda - 0{,}000{,}003\, h, \qquad (84)$$

where *g* is the value of *g* at any point the latitude of which is λ, and its height above the sea *in centimeters* is *h* (978.07 is the value taken for *g* at level of the sea at the equator).

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* See Mendenhall, Report of U. S. Coast and Geodetic Survey, 1891, II. p. 504; also G. R. Putnam, Bulletin, vol. xiii., Philos. Society of Washington, p. 31.

Observations of the value of g have been made in many places by the method of reversible pendulum. The results obtained vary from those given by Equation (84), on account of the lack of homogeneity of the earth. The proximity of mountain chains and of the deep sea are the chief sources of irregularity, and it is at stations where such causes of variation occur that the largest discrepancies have been noted.

The following are some of the results of the experimental determination of g:

TABLE I.

Locality.	Latitude.	Longitude.	Elevation.	Value of G (not reduced to sea-level).
Boston, Mass.	42° 21′ 33″	71° 03′ 50″	22 meters	980.382
Philadelphia, Pa. . . .	39° 57′ 06″	75° 11′ 40″	16 “	980.182
Washington, D. C. . .	38° 53′ 20″	77° 01′ 32″	10 “	980.100
Ithaca, N. Y.	42° 27′ 04″	76° 29′ 00″	247 “	980.286
Cleveland, O.	41° 30′ 22″	81° 36′ 38″	210 “	980.227
Cincinnati, O.	39° 08′ 20″	84° 25′ 20″	245 “	979.990
Terre Haute, Ind. . .	39° 28′ 42″	87° 23′ 49″	151 “	980.058
Chicago, Ill.	41° 47′ 25″	87° 36′ 03″	182 “	980.264
St. Louis, Mo.	38° 38′ 03″	90° 12′ 13″	154 “	979.987
Kansas City, Mo. . . .	39° 05′ 50″	94° 35′ 21″	278 “	979.976
Denver, Col.	39° 40′ 36″	104° 56′ 55″	1638 “	979.595
San Francisco, Cal. . .	37° 47′ 00″	122° 26′ 00″	114 “	979.951
——	——	——	——	——
Greenwich	51° 29′ 00″	0° 00′ 00″	47 “	981.170
Paris	48° 50′ 11″	2° 20′ 15″	72 “	980.960
Berlin	52° 30′ 16″	13° 23′ 44″	35 “	981.240
Vienna	48° 12′ 35″	16° 22′ 55″	150 “	980.852
Rome	41° 53′ 53″	12° 28′ 45″	53 “	980.312
Hammerfest	70° 40′ 00″	22° 38′ 00″	——	982.580

CHAPTER VI.

ELASTICITY; FRICTION OF SOLIDS.

116. Stress and strain. — When external forces act upon a body and tend to change its shape, internal forces are brought into action between contiguous parts of the body throughout. This force action between contiguous parts of a body is called *stress.*

A body under stress is always distorted; such distortion is called *strain.*

STRAIN.

117. Principal stretches of a strain.* — A small spherical portion of a body becomes an ellipsoid when the body is strained.†

A distortion which changes a sphere into an ellipsoid is always equivalent to three mutually perpendicular **stretches,** parallel, if there has been no rotation, to the axes of the resulting ellipsoid.‡ A stretch here signifies a uniform increase (or

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* Sometimes called principal dilatations.

† *Proof.* — Let the origin of co-ordinates be chosen at the center of the small spherical portion of the body. Let $x^2 + y^2 + z^2 = r^2$ be the equation to the surface of this small sphere. Let x', y', and z' be the co-ordinates of the point xyz after strain. If there is no rupture in the neighborhood, x', y' and z' are continuous functions of x, y, and z, and may each be expanded in a series of ascending powers of x, y, and z. Since x, y, and z are very small, squares and higher powers may be neglected, giving

$$x' = a_1x + b_1y + c_1z,$$
$$y' = a_2x + b_2y + c_2z,$$
$$z' = a_3x + b_3y + c_3z,$$

in which the a's, b's and c's are constants. These equations transform the equation to the small sphere into an equation still of the second degree, which can be shown to be the equation to an ellipsoid.

‡ This proposition is a generalization of the one to the effect that the distortion which changes a circle to an ellipse is always equivalent to two mutually perpendicular stretches.

decrease) in the linear dimensions of a body in a certain direction. Let $1 : 1+\alpha$ be the ratio of increase of linear dimensions of a body in the direction of a stretch. The quantity α is adopted as the **measure** of the stretch. The three stretches of a strain are here represented by α, β, and γ.

118. The strain ellipsoid. — The semiaxes of the actual ellipsoid into which a small spherical portion of radius, r, of a body is distorted by a strain, are $(1+\alpha)r$, $(1+\beta)r$, and $(1+\gamma)r$. An ellipsoid, similarly related to a sphere of *unit* radius, having semiaxes $(1+\alpha)$, $(1+\beta)$, and $(1+\gamma)$ parallel to the respective axes of the small ellipsoid, is called the **strain ellipsoid.** This strain ellipsoid in its relation to the unit sphere is a complete geometrical representation of the distortion of the small spherical portion of the body.

Axes of strain. — The directions of the stretches of a strain or of the axes of the strain ellipsoid are called the *axes* of the strain.

119. Homogeneous strains. — A body is said to be **homogeneously** strained when the strain ellipsoids in all parts of a body are similar and similarly placed. A stretched wire is homogeneously strained. A bent beam and a twisted wire are examples of non-homogeneous strain.

Remark. — All that is here said of stress and strain applies either to finite portions of bodies homogeneously stressed or strained, or to the smaller parts of bodies non-homogeneously stressed or strained.

120. Distortion of a cubical portion of a body. — A cubical portion of a body becomes, after the strain, a parallelopiped. If the edges of the cube are parallel to the axes of the strain, the parallelopiped will be rectangular, otherwise not.

121. Specification of strain. — A strain unaccompanied by rotation is completely specified by its stretches α, β, and γ, together with the three angles necessary to specify their directions.

STRESS.

122. Stress on a section. — Imagine a small plane area ΔS, called a **section,** in the interior of a body under stress. The portions of the body on the two sides of this section act on each other in a certain direction with a force ΔF. The quotient $\frac{\Delta F}{\Delta S}$ is called the **stress on the section.** When ΔF is perpendicular to ΔS, the stress is called a **normal** stress. When ΔF lies in the plane of ΔS, the stress is called a **tangential** stress. When ΔF is in any other direction, the stress on the section can be resolved into normal and tangential components.

123. Homogeneous stress. — A body is said to be under homogeneous stress when the stress on all parallel sections is the same in magnitude and direction. The stress in a stretched wire is homogeneous; the stress in a twisted wire is non-homogeneous.

124. Principal sections of a stress; Pulls of a stress. — The conditions of equilibrium of a small portion of a body, as expressed by equations (52), lead * to the result that at a point in a body under stress there must be three mutually perpendicular sections upon which the stress is normal. That is, the stress in the neighborhood of a point in a body consists of three mutually perpendicular *pulls.*† These three sections are called the **principal sections** of the stress, and the three pulls are called the **elements** of the stress.

125. Axes of a stress. — The directions of the three pulls of a stress are called the axes of the stress.

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* For details of proof see Ency. Brit., 9th ed., *Theory of Elasticity*, Chap. III. — Sir W. Thomson.

† The word **pull** here signifies a normal stress on a section that is **force per unit area,** a **push** being considered a negative **pull.**

126. Specification of stress. — A stress is completely specified by its three **pulls,** together with the three angles necessary to specify their directions.

127. Problem. — To find the stress on any section when the stress is specified as above, and the direction of the normal to the given section is specified. Let *XYZ* (Fig. 65) be the axes of the stress; *f*, *g*, and *h* the pulls parallel to *X*, *Y*, and *Z* respectively; *XYZ* the given section of area *q*. Let *l*, *m*, and *n* be the angles between the normal to the given section, and the axes *X*, *Y*, and *Z* respectively. Consider the material inside the tetrahedron *XYZO*. It is in equilibrium; therefore (equations 49) the forces acting on this material across the faces *YOZ*, *XOZ*, and *XOY*, are equal and opposite to the respective components of the force acting across the given section *XYZ*. Dividing these components of the force acting across *XYZ* by *q*, we have the components of the required stress on *XYZ*. The areas of *YOZ*, *XOZ*, and *XOY* are *q* cos *l*, *q* cos *m*, and *q* cos *n* respectively. The forces acting across these faces are $-fq \cos l$, $-gq \cos m$, and $-hq \cos n$, respectively. Since *f*, *g*, and *h* are *pulls*, these forces are considered negative. They are in the negative directions of the respective axes. The components of the stress on the section *XYZ* are, therefore, **f cos l, g cos m,** and **h cos n.**

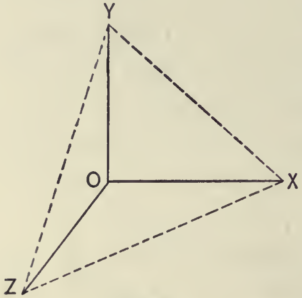

Fig. 65.

STRESS; STRAIN (RELATIONS OF).

128. Isotropy and ælotropy. — A body is said to be **isotropic** when it has identically the same physical properties in all directions; otherwise it is said to be **ælotropic.** Annealed glass, cast metal, liquids, gases, etc., are isotropic; rolled metal, wood, many crystals, etc., are ælotropic.

The axes of stress and strain coincide in isotropic bodies.

This is not true in ælotropic bodies. The discussion of the theory of the elasticity of ælotropic bodies is beyond the scope of this text.

129. Hooke's law. — The ratio, for a given substance, of a given **type*** of stress and strain is a constant. This ratio is called the **elastic modulus** of the substance for the given type of stress and strain.

Remark. — For very small stresses and strains Hooke's law is necessarily true. As expounded by Hooke, however, it was thought to hold with accuracy throughout a wide range of stress and strain. See Art. 156.

TYPES OF STRESS AND STRAIN.

130. Hydrostatic pressure. — A hydrostatic pressure is a stress having its three pulls equal. The common value of these pulls is adopted as the *measure* of the hydrostatic pressure.

131. Pascal's principle. — In a substance under hydrostatic pressure the stress on all sections is normal and of the same value.

Proof. — In the problem of Art. 127 put $f = g = h = p$. Then the components of the stress on the section XYZ become $p \cos l$, $p \cos m$, and $p \cos n$, which are evidently the components of a

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* Stresses and strains, inasmuch as they depend upon a number of independent numerical specifications (see Arts. 121 and 126), are called **complex** quantities. Such quantities do not in general conform to the principles of ordinary algebra. If it is desired to treat complex quantities of a certain kind by means of ordinary algebra, it is necessary to adopt a number of simple **ground forms** or **types** of that quantity. Any value of the quantity may then be **resolved** into its **ground form** components, each of which is capable of specification by a single numerical specification, and conforms in consequence to the principles of ordinary algebra. The points here raised will become clear in the following discussion of **types of stress and strain.** Until this discussion is carefully studied, the above statement of Hooke's law will not be intelligible, because the student cannot previously grasp the conception of the ratio $\frac{\text{stress}}{\text{strain}}$. [See the works of Tait, Hamilton, Grassmann, and Klein on Complex quantity.]

stress equal to p and which makes angles l, m, and n with the axes X, Y, and Z respectively, as does the normal to the section.

Corollary. — A stress such that on no section is there any tangential stress is a hydrostatic pressure.

132. Isotropic strain. — Isotropic strain is a strain having its three stretches α, β, and γ equal. It is the strain which in isotropic bodies accompanies hydrostatic pressure.

133. Measure of isotropic strain. — Isotropic strain could be specified by the common value, i, of α, β, and γ, but it is customary to make use of the ratio $\frac{\Delta v}{v}$ as its *measure;* Δv being the change in volume of a portion of the body of which the initial volume is v.

134. Proposition. — The change of volume per unit volume $\left(\frac{\Delta v}{v}\right)$ of a body under isotropic strain is equal to three times the *stretch* (i) of the body.

Proof. — Consider a cubical portion of the body, the edges of the cube being of length l. After strain the length of edge becomes $l(1 + i)$. We therefore have $v = l^3$ and $v + \Delta v = l^3(1 + i^3) = l^3 + 3il^3 + \cdots$. Discarding terms in i^2 and i^3, we find directly

$$\frac{\Delta v}{v} = 3i. \qquad (85)$$

135. Bulk modulus.[*] — The constant ratio $\frac{\text{hydrostatic pressure}}{\text{isotropic strain}}$ for a given substance is called the **bulk modulus** of that substance. This ratio is

$$V = \frac{p}{\frac{\Delta v}{v}}, \qquad (86)$$

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[*] The reciprocal of bulk modulus is sometimes called the compressibility of a substance.

in which V is the bulk modulus of a substance, v its volume, and Δv the change in its volume produced by a hydrostatic pressure p.

136. Shearing stress. — A shearing stress is a stress having two of its pulls equal and opposite, and the third equal to zero. The common numerical value of the two pulls is adopted as the measure of a shearing stress.

Consider a cubical portion $ABCD$ of a body under shearing stress, the faces of the cube being principal sections of the stress. The stress on the faces AB and CD (Fig. 66) is a *push* S, on the faces AC and BD an equal *pull*. On the faces of the cube, parallel to the paper, the stress is zero. The plane $ABCD$, of the paper, is called **the plane of the shearing stress.**

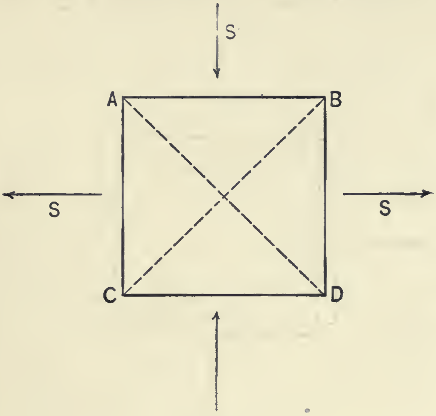

Fig. 66.

137. Proposition. — The stress on the diagonal sections AD and CB (Fig. 66) is purely tangential and equal to S.

Proof. — Let l be the length of an edge of the cube. Consider the portion ABD of the cube. It is in equilibrium, and the resultant of the forces acting across the faces AB and BD, ignoring sign, is equal to, and has the same line of action as, the force acting across the face *AD*. The area of each *AB* and *BD* is l^2, and of *AD* is $l^2\sqrt{2}$. The resultant of the forces Sl^2 and Sl^2, which act across *AB* and *BD* is a force **lying in the line AD**, and equal to $Sl^2\sqrt{2}$, which, divided by the area of *AD*, gives *S*. Q.E.D.

138. Shearing strain. — A shearing strain is a strain having two of its stretches equal and opposite, and the third equal to zero. It is the strain which in isotropic bodies accompanies shearing stress.

139. Measure of shearing strain; Angle of shear. — Let the heavy line square (Fig. 67) represent a cubical portion of a body which is changed under the shearing strain into a rectangular parallelopiped represented by the dotted rectangle. Dimensions perpendicular to the plane of the paper remain unaltered. The dotted rhombus is a figure which was square in the unstrained cube. The angle $\phi = \frac{\pi}{2} - \theta$ is called the **angle of the shearing strain** and is adopted as the measure of the strain.

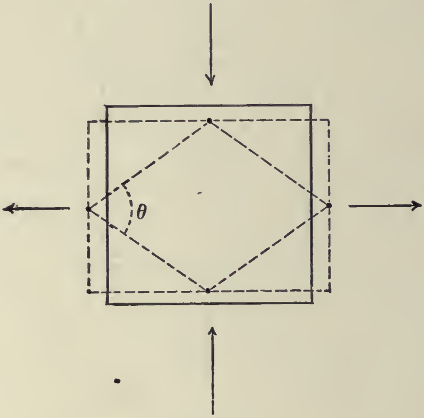

Fig. 67.

140. Proposition. — The angle ϕ of a shearing strain is equal to twice the stretch, α, of the strain.

Proof. — Let the rhombus *eCDB* (Fig. 68) be the dotted rhombus of Fig. 67, *turned* so that its side *DB* coincides with the side of the square from which it was distorted by the strain. Let l be the length of *AD*. Then *the length of the diagonal AB is* $l\sqrt{2}$. Since ϕ is small, $Ae = l\phi$, and since *Aef* is sensibly a right triangle with equal legs, *Af* is equal to $\frac{Ae}{\sqrt{2}}$. Further, the shortened diagonal *eB* of the rhombus is sensibly equal to $AB - Af$, therefore *the length of the shortened diagonal eB of the rhombus is* $l\sqrt{2} - \frac{l\phi}{\sqrt{2}}$. From the definition of stretch we have $1 - \alpha = \frac{eB}{AB}$. Substituting in this the above values of *AB* and *eB*, we have

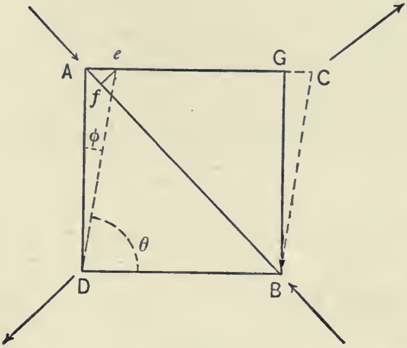

Fig. 68.

$$\alpha = \tfrac{1}{2}\phi. \tag{87}$$

141. Slide modulus.* — The constant ratio $\frac{\text{shearing stress}}{\text{shearing strain}}$ for a given substance is called the **slide modulus** of that substance. It is defined by the equation

$$n = \frac{S}{\phi}, \tag{88}$$

in which n is the slide modulus of a substance, and ϕ is the angular distortion produced by a shearing stress S.

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* Often called **coefficient of simple rigidity.**

142. Simple and compound stresses. — **Hydrostatic pressure** and **shearing stress** being chosen as ground-forms or *types* of stress, are called *simple stresses.*[*] Any stress which is neither a hydrostatic pressure nor a shearing stress can be resolved in general into a hydrostatic pressure and two shearing stresses of which the planes are at right angles. These are called the **simple stress components** of the given stress, and the latter is called a **compound** stress.

Remark. — Identical statements hold for **simple** and **compound strain.**

143. Longitudinal stress. — A longitudinal stress is a stress having one pull finite and the other two equal to zero. The value of the finite pull is adopted as the measure of the stress. The direction of the finite pull is the only axis of the stress which is determinate. It is called *the axis* of the stress.

144. Longitudinal strain. — A longitudinal strain is the strain which in isotropic bodies accompanies longitudinal stress. All three stretches of a longitudinal strain are *finite.* The stretch in the direction of the axis is adopted as the measure of the strain. The other two stretches are equal. They are always opposite in sign to the axial stretch and bear to that stretch a constant ratio, for a given substance, called *Poisson's ratio.*

145. Stretch modulus.[†] — The constant ratio

$$\frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

for a given substance is

$$E = \frac{P}{\beta}, \tag{89}$$

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[*] **Hydrostatic pressure** and **shearing stress** are chosen as simple types, because they are **Orthogonal**; that is, no work is done upon a body by a hydrostatic pressure when the body suffers shearing strain, nor by a shearing stress when the body suffers isotropic strain. See Article, *Elasticity, Theory of.* Ency. Brit., 9th ed. Sir W. Thomson.

[†] Often called Young's modulus.

in which β is the stretch produced by a pull P in its own direction. The ratio E is called the *stretch modulus* of the substance.

146. The experimental determination of Poisson's ratio. — The change which the contents of a tube, of inner diameter d (Fig. 69) and length C, suffers when stretched is obviously identical with the change of volume which a rod, of the same diameter (d) and length, and composed of the same material as the tube, undergoes when equally stretched. This fact is used to determine Poisson's ratio.

An excellent material for such an experiment is glass. A tube with strong walls is closed at one end, and after it has been nearly filled with water or mercury, the other end is drawn out into a capillary neck. The neck is bent (Fig. 70), to facilitate

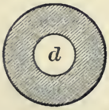

Fig. 69.

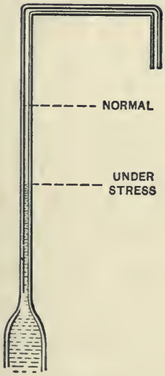

Fig. 70.

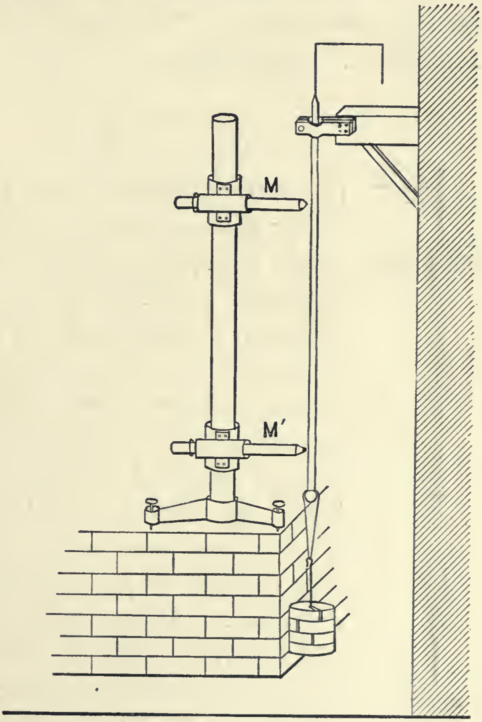

Fig. 71.

further filling, which is accomplished by successive heatings and coolings of the tube. After filling until the level of the contained liquid is within the capillary neck, the tube is mounted vertically as shown in Fig. 71, and weights are applied to stretch it.

An easily measurable downward movement of the liquid in the neck indicates increase in the volume of the tube when stretched. By calibration of the total content and of the contents of the capillary neck, per centimeter length, the change of volume corresponding to each pull may be determined, whence the lateral stretch (γ) is readily computed. By observing at the same time the elongation of the tube, for which purpose the reading microscopes M, M' are used, the longitudinal stretch β is determined. The quantity

$$\sigma = \frac{\gamma}{\beta}, \tag{90}$$

(see further Arts. 144 and 147) is Poisson's ratio.

For glass, σ, has almost precisely the value $(\frac{1}{3})$ which Poisson himself supposed it to have in the case of all solids.

Where metals are to be tested, it is necessary to modify the apparatus, giving it the form of a tube with heavy caps. Through the upper one, by which also the tube is fastened, a capillary neck of glass is inserted. The tube is filled with liquid, the sinking of the level of which within the neck where weights are applied to the lower cap, indicates the increase of volume as already described. Figure 72 shows this apparatus, which is that used by Wertheim* in his well-known experiments. Wertheim deemed his results confirmatory of Poisson's theoretical conclusion, *i.e.* that σ is a constant and equal to $\frac{1}{3}$; but it

Fig. 72.

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* Wertheim, Annales de Chimie et de Physique, (3) 12, p. 385 (1844).

has since been shown by means of numerous experiments, some of which are analogous to the above and some of which are based upon the relations discussed in Art. 147, that Poisson's ratio varies with the material and with its molecular condition.

147. Relation between bulk, slide, and stretch moduli and Poisson's ratio. — Longitudinal stress can be resolved into simple stress components, and the strain accompanying each of these components V and n, being known, can be calculated from equations (86) and (88). These strains superposed give the actual longitudinal strain produced by the given longitudinal stress. Therefore both the stretch modulus and Poisson's ratio of a substance must admit of calculation in terms of the bulk and slide moduli of the substance; or, in other words, these four quantities must satisfy two independent equations. The following derivation of these two equations given below follows the above outline.

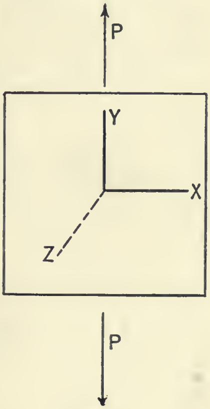

Fig. 73.

Consider a cubical portion of a body under longitudinal stress P; for example, a portion of a stretched rod, as shown in Fig. 73. Imagine the pull P to consist of three equal parts, $\frac{1}{3}P$, and imagine the lateral faces of the cube to be subjected each to a *pull* $\frac{1}{3}P$ and a *push* $\frac{1}{3}P$. (This supposition is allowable since the pull and push are together equal to zero.)

Since the pull $\frac{1}{3}P$ in all directions constitutes a negative hydrostatic pressure p, we have

$$p = \tfrac{1}{3}P. \qquad (a)$$

The pushes on the x-faces, together with the pull $\frac{1}{3}P$ on the y-faces, constitute a shearing stress

$$S = \tfrac{1}{3}P, \qquad (b)$$

of which the plane is xy.

The pushes on the z-faces together with the remaining pull on the y-faces constitute another shear S' of which the plane is yz; so that we have

$$S'=\tfrac{1}{3}P. \tag{c}$$

The above are the simple stress components of the longitudinal stress. The hydrostatic pressure (a) produces an isotropic strain of which the stretch is i in all directions and positive. From equations $3i=\frac{\Delta v}{v}$ (85), $V=\frac{p}{\frac{\Delta v}{v}}$ (86), and $p=\frac{1}{3}P$ (a), we have

$$i=\frac{P}{9V}. \tag{d}$$

Each shearing stress (b) and (c) produces a shearing strain of which the stretches are in each case $+\alpha$ and $-\alpha$. From equations $2\alpha=\phi$ (87), $n=\frac{S}{\phi}$ (88), and $S=\frac{1}{3}P$ (b), we have

$$\alpha=\frac{P}{6n}. \tag{e}$$

Since each shearing strain contributes a stretch α in the y direction, the total stretch, β, in that direction is

$$\beta=2\alpha+i. \tag{f}$$

Further, the shearing strains contribute stretches $-\alpha$ in the x and z directions, respectively; so that the total lateral stretch γ is

$$\gamma=i-\alpha. \tag{g}$$

By definition Poisson's ratio, σ, is

$$\sigma=\frac{\gamma}{\beta}. \tag{90 bis}$$

The equation for stretch modulus is

$$E=\frac{P}{\beta}. \tag{89 bis}$$

Substituting values of i and α from (d) and (e) in (f) and (g) and the resulting values of β and γ in (90) and (89), we have

$$\sigma=\frac{3V-2n}{6V+2n}; \tag{91}$$

$$E=\frac{9Vn}{3V+n}. \tag{92}$$

EXAMPLES OF STRESS AND STRAIN.

148. Heavy liquid in a vessel. — A horizontal section (SS' of Fig. 74) of area a at a distance x beneath the surface of a liquid of density d is acted upon by a force $axdg$ due to the weight of the liquid above it, g being the acceleration of gravity. The stress on the section is therefore xdg, and this must be the hydrostatic pressure p at the point, so that

$$p=xdg. \tag{93}$$

If the pressure at the surface of the liquid is p_0, then

$$p=p_0+xdg. \tag{94}$$

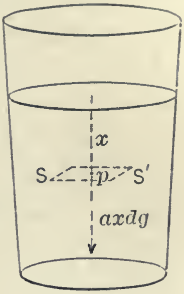

Fig. 74.

Thus the non-homogeneous hydrostatic pressure is specified at all points in the liquid in terms of the one co-ordinate x.

149. Stretched wire. — A wire of sectional area q, held tense by a force F, is under homogeneous longitudinal stress P such that

$$P=\frac{F}{q}. \tag{95}$$

Let l be the initial length of the wire, and $l+\Delta l$ its length under tension, then

$$l : l+\Delta l = 1 : 1+\beta;$$

whence $\beta=\frac{\Delta l}{l}$. Substituting this value of β, and the above value of P in $E=\frac{P}{\beta}$ (89), we have

$$E=\frac{Fl}{q\Delta l}. \tag{96}$$

This equation enables the calculation of E when F, l, q, and Δl have been observed.

The wire to be experimented upon is fastened at one end to a wall-bracket (Fig. 75) or other firm support, and a weight is applied at the free end. The wire

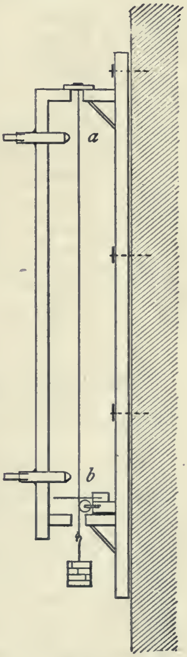

Fig. 75.

thus rendered vertical and tense is further stretched by the addition of small weights of known mass.

Reading microscopes are focused upon points a and b near the upper and the lower ends of the wire. The difference in the movement of the two measures the elongation Δl. The quantity l, which is the distance ab, is found by direct measurement; while q is computed from the diameter, or from the density of the material and the weight (per meter) of the wire.

If F be expressed in dynes, l and Δl in centimeters, and q in square centimeters, equation 96 gives the modulus in c. g. s. units. The following table contains the value of the stretch modulus for several materials.

TABLE II.

Stretch Modulus of Various Materials.

Brass	0.83×10^{12}	to 0.97×10^{12}
Copper	0.98	to 1.0
Gold	0.52	to 0.79
Iron	1.7	to 2.4
Lead	0.15	to 0.18
Platinum	1.5	to 1.7
Silver	0.7	to 0.71
Zinc	0.85	
Glass	0.66	to 0.78
Birch	0.091	
Oak	0.092	
Poplar	0.05	
Spruce	0.11	

150. Bent beam.[*] — Consider such a portion of a slightly bent beam (Fig. 76) as subtends a small angle θ from its center of

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[*] This discussion of a bent beam, although the one usually given in elementary treatises on Mechanics, which is accurate enough for most purposes, is incorrect. This is evident when we consider that the assumed strain, having no lateral stretch, is not a true longitudinal strain. There is nothing *in itself* which shows the assumed distortion of a beam to be impossible, but if the *pulls* at each point necessary to produce the assumed simple elongation be calculated, it will be found that the *vertical* pulls cannot balance each other in the different parts of the beam, and the particles of the beam will not therefore be in equilibrium. To calculate the *pulls* necessary to produce simple elongation, resolve the simple elongation into an isotropic strain and two shearing strains; calculate the corresponding stresses and superpose them.

curvation C. All the filaments of the beam in the short portion are assumed to be circular and concentric. The filaments in the upper part of the beam are stretched, those in the lower part are shortened. An intermediate filament retains its initial length. This filament marks the so-called median line of the beam represented in the figure by the dotted line. Let R be the radius of curvature of the median line; then $R\theta$ is the initial length of all filaments between AC and BC. The length of the filament at a distance x above the median line is $(R+x)\theta$.

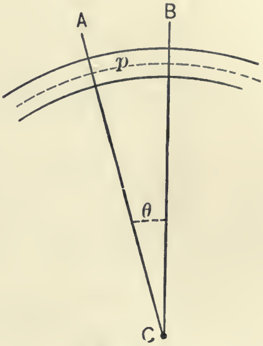

Fig. 76.

Therefore $$1 : 1 + \beta = R\theta : (R + x)\theta,$$

whence $$\beta = \frac{x}{R}, \tag{97}$$

in which β is the stretch of the filament.

Substituting this value of β in (89) and solving for P, we have

$$P = \frac{Ex}{R}. \tag{98}$$

In this equation P is the longitudinal stress in that part of the beam distant x above the median line.

151. Proposition. — The total force action across a complete section of a bent beam is a torque T about a line p (Fig. 76) perpendicular to the plane of the paper, such that

$$T = \frac{1}{12}\frac{a^3bE}{R}, \tag{99}$$

in which a is the depth of the beam and b is its width.

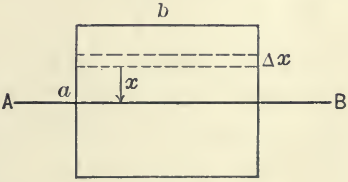

Fig. 77.

Proof. — Let Fig. 77 represent a cross-section of the beam.

The force ΔF acting across the area $b\Delta x$, of the narrow dotted strip, is $\Delta F = Pb\Delta x$, or using value of P from (98),

$$\Delta F = \frac{Eb}{R} x \Delta x.$$

The torque ΔT of this force about the line AB is

$$\Delta T = \frac{Eb}{R} x^2 \Delta x,$$

whence the torque across the complete section is

$$T = \frac{Eb}{R} \int_{-\frac{a}{2}}^{+\frac{a}{2}} x^2 dx = \frac{1}{12} \frac{a^3 b E}{R}. \qquad (100)$$

152. Twisted wire. — Consider those portions of a twisted wire which lie in the *surface of a cylinder* coaxial with the wire and of radius r — less than the radius of the wire. Let l be the length of the wire, and θ the angle through which one end of it is turned. Let Figs. 78 and 79 represent the development of this cylindrical surface, Fig. 78 before twisting and Fig. 79 after twisting. The line AB is brought into the position $A'B'$, and the small square is distorted into the rhombus by the twist. The portions of the wire under consideration are thus subjected to a

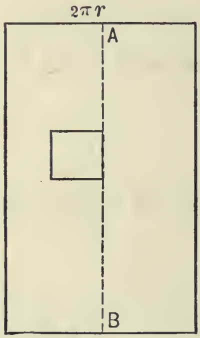

Fig. 78.

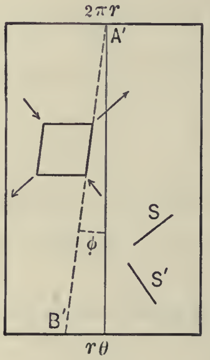

Fig. 79.

shearing strain, of which the angle ϕ, as is evident from the figure, is

$$\phi = \frac{r\theta}{l}. \qquad (101)$$

Substituting this value of ϕ in $n = \frac{S}{\phi}$ (88), and solving for S, we have

$$S = \frac{nr\theta}{l}, \qquad (102)$$

in which S is the shearing stress in the portion of the wire under consideration. Comparing Fig. 79 with figures in articles 136, 139, and 140, we see that the stress at a point in a twisted wire is such that (1) on all sections perpendicular to the axis of the wire the stress is purely tangential, and perpendicular to the radius; (2) on all sections of which the planes include the axis of the wire the stress is purely tangential and parallel to the axis of the wire; (3) on all sections which are perpendicular to a radius the stress is zero; (4) on such sections as S (Fig. 79) the stress is a push, and (5) on such sections as S' (Fig. 79) the stress is a pull. In every case the value of the stress is that given by equation (102).

153. Proposition. — The total force action across a complete section of a twisted wire is a torque T about the axis of the wire such that

$$T = -\frac{\pi n \rho^4 \theta}{2l}, \qquad (103)$$

in which ρ is the radius of the wire.

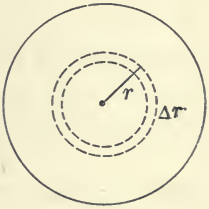

Fig. 80.

Proof. — Let Fig. 80 represent a sectional view of the wire. Consider the annular portion of width Δr and radius r shown by the dotted lines. The force action per unit area across this annulus is $\frac{rn\theta}{l}$ and **perpendicular** to r at each

point, the area of the annulus being $2\pi r\Delta r$. The total torque action across it is

$$\Delta T = 2\pi r\Delta r \cdot \frac{rn\theta}{l} \cdot r,$$

whence

$$T = \frac{2\pi n\theta}{l}\int_0^\rho r^3\,dr = \frac{\pi n\rho^4\theta}{2l}.$$

The negative sign is written for the reason that T and θ are opposite in sign.

154. Constant of torsion of a wire (sometimes called *moment of torsion*). — The quantity $\frac{\pi n\rho^4}{2l}$, being the quantity which multiplied by the angle of twist of a wire gives the twisting torque, is called the constant of torsion of the wire.

For the determination of this constant, the principle of the torsion pendulum, already considered (see Arts. 105 *et seq.*), is used. A cylindrical or spherical weight, the moment of inertia of which may be computed from its dimensions and mass (see Art. 107), is suspended from the wire to be tested and the period of the torsion pendulum thus formed is determined.

155. The torsion pendulum may likewise be employed in the determination of the *slide modulus* (n). See Art. 141.

Comparing

$$T = -\frac{\pi n\rho^4}{2l}\cdot\theta \text{ (103) with } T = -b\theta \text{ (70) and } \frac{4\pi^2 K}{\tau^2} = b \text{ (71),}$$

we have

$$\frac{4\pi^2 K}{\tau^2} = \frac{\pi n\rho^4}{2l}, \tag{104}$$

in which K is the moment of inertia, and τ the period of vibration of a body hung by the wire. This equation enables the calculation of n when the other quantities have been determined.

156. In speaking of the elastic properties, bodies are said to be *perfectly elastic* when they return, after the removal of stress, to precisely their former configuration. Solids are perfectly elastic only when the distortion to which they have been subjected is slight.* A body is said to be highly elastic when it possesses a large modulus.

When the distortion to which a body is subject is considerable enough to render its failure to return measurable, it is said that the *limit of elasticity* has been passed.

If the stress, of whatever type, be further increased, a point is finally reached at which rupture of the distorted body occurs. This is termed the *breaking stress*.

157. The experimental investigation of the elastic properties of matter has brought to light many phenomena, which are not considered in the preceding discussion. Thus in the case of all finite stresses, the strain depends upon the length of time during which the forces are allowed to act.

These progressive effects have been followed over periods of many years, and it has been found, for example, that a longitudinal pull, far below the breaking stress, will frequently produce rupture if applied for a sufficient length of time.

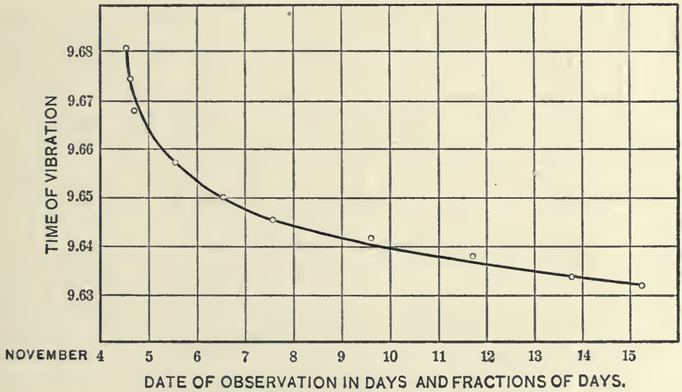

Fig. 81.

Bodies which have been under continued stress, on the other hand, require time, and often a very long time, in which to

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* See J. O. Thompson, American Journal of Science, Vol. 43, p. 32.

resume, by continuous readjustment, their ultimate unstrained configuration. Even the strains which are brought into existence in such a process as the drawing of wire have been found to exert a temporary but long-continued influence upon the elasticity of the metal. In the case of a phosphor bronze wire, for example, the constant of torsion was found to change from day to day, at first rapidly, then more and more gradually as the age of the wire diminished. The effect was distinctly traceable, although greatly reduced after the lapse of four years. How regular the law of its decadence was, may be seen from Fig. 81, in which the period of oscillation is shown as a function of the time.

158. Temperature also affects the elasticity of solids, as does indeed every change in the molecular arrangement of a body. Almost without exception the effect of a rise of temperature is to *lower the modulus*. Thus the bronze wire, to which reference has just been made, when carrying a certain cylindrical weight had a period the variation of which between 3° and 10° is indicated by the curve in Fig. 82.

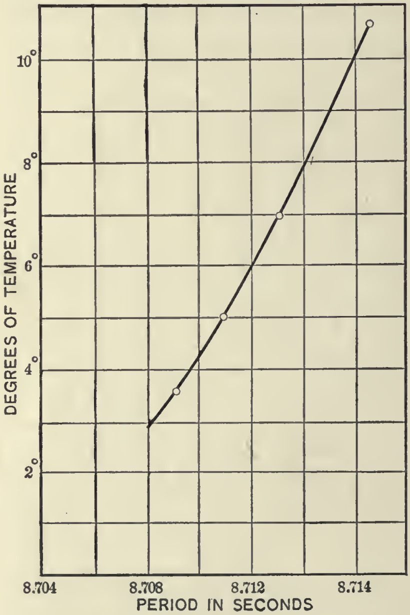

Fig. 82.

The value of the slide modulus is inversely proportional to the square of the period of vibration, see Art. 155, so that the curve between slide modulus and temperature, derived from the curve in Fig. 82, is similar to the curve in Fig. 83. The mean percentage change of slide modulus per degree centigrade (temperature coefficient), as derived from the data shown in Fig. 82, is .00045.

An analogous example, in the case of the stretch modulus, is shown in (Fig. 83), which gives the curve of variation of stretch-modulus of a steel wire with temperature between 20° and 160°.* The temperature coefficient in this case is .0005.

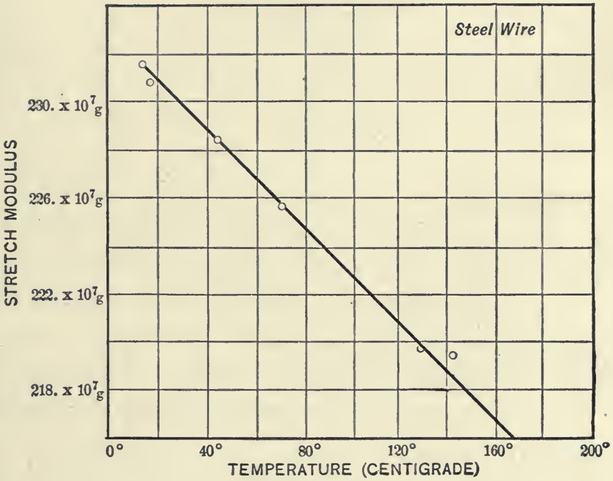

Fig. 83.

Potential Energy of Strain.

159. Of longitudinal strain. — Consider a cubical portion, edge l, of a substance under longitudinal stress P, stretched to length $l+m$ in the direction of P; m being small. From the definition of stretch we have $1 : 1+\beta = l : l+m$, whence $\beta = \frac{m}{l}$ or $d\beta = \frac{dm}{l}$; β being the measure of the longitudinal strain. Let P increase slightly, causing a further increase dm in the length of the cube, and making it, in fact, equal to $l+m+dm$. The work dW, done in producing this increment, dm, of length, is Pl^2dm, for the area across which the *pull* P acts is sensibly

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* See paper of Miss M. C. Noyes, Physical Review, Vol. II., p. 277.

l^2, and the stretching *force* Pl^2 acts through the distance dm. Therefore $dW = Pl^2 dm = Pl^3 \frac{dm}{l} = Pl^3 d\beta$. In this expression write v (volume of cube) for l^3 and substitute the value of P from $E = \frac{P}{\beta}$ (89), whence $dW = vE\beta d\beta$. Since v is sensibly constant, and W is zero when β is zero, we have by integration

$$W = \tfrac{1}{2} vE\beta^2, \tag{105}$$

or substituting $\frac{P}{\beta}$ from (59) for E,

$$W = \tfrac{1}{2} vP\beta. \tag{106}$$

160. Of isotropic strain. — Consider a cube under hydrostatic pressure p, and let i be the stretch in all directions of the accompanying strain. In a manner similar to the method of Art. 159, it can be shown that $\frac{1}{2} vpi$ (compare equation 106) is the work done by the force acting across each pair of faces of the cube while the strain is being established. Therefore the total work done in establishing the strain is $\frac{3}{2} vpi$. Substituting $\frac{\Delta v}{v} = 3i$ from (85), we have

$$W = \tfrac{1}{2} vp\left(\frac{\Delta v}{v}\right), \tag{107}$$

or, substituting the value of p from (86), we have

$$W = \tfrac{1}{2} vV\left(\frac{\Delta v}{v}\right)^2. \tag{108}$$

161. Of shearing strain. — Consider a cube under shearing stress S, and let $+a$ and $-a$ be the stretches of the shearing strain. In a manner similar to the method of Art. 159 it can be shown that $\frac{1}{2} vSa$ (compare equation 106) is the work done by each pull of the shearing stress while the strain is being established. Therefore the total work done in establishing the strain is $\frac{1}{2} vS\, 2a$. Substituting $\phi = 2a$ from (87), we have

$$W = \tfrac{1}{2} vS\phi; \tag{109}$$

or substituting the value of S from equation (88), we have

$$W = \tfrac{1}{2} vn\phi^2. \tag{110}$$

Remark. — The potential energy per unit volume at a point in a strained body is then equal to one-half the product of the stress into the strain, or to one-half the product of the modulus into the square of the strain.

162. Friction. — The forces which oppose the tangential motion of bodies in contact give rise to the phenomena of friction. These phenomena are very complex on account of the extreme variety of the ruling conditions, but the following statement (Coulomb's law of friction) covers the ground in the case of solids sufficiently well for most purposes.

Coefficient of friction. — The tangential force necessary to cause the sliding of two bodies with their surfaces in contact, bears an approximately constant ratio to the normal force pushing the surfaces together. This ratio is called the **coefficient of friction.** It is sensibly independent of the area of contact and of the velocity of sliding, but depends upon the character of the surfaces and upon the nature of the bodies, *e.g.* upon their hardness. This law of friction was first noted by Coulomb (1781).

163. Angle of friction. — Consider two bodies with flat surfaces in contact. Let the force F (Fig. 84), with which they act on each other across the surface of contact, be inclined at an angle ϕ to the normal to that surface. Then $F \cos \phi$ is the normal force pushing the surfaces together, and $F \sin \phi$ is the tangential force tending to cause sliding. When the ratio $F \sin \phi \div F \cos \phi$ becomes equal to the coefficient of friction, μ, of the surfaces, sliding will take place, therefore the maximum value of the angle ϕ is such that

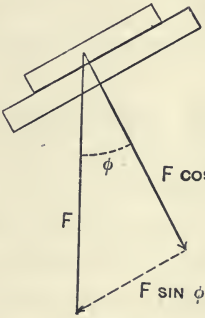

Fig. 84.

$$\tan \phi = \mu. \tag{111}$$

The angle ϕ is called the **angle of friction,** sometimes also the angle of repose, of the two surfaces.

164. Limitations to Coulomb's law of friction. — The friction between two surfaces is, no doubt, due in part to a continual interlocking and release of fine protuberances on the sliding surfaces and in part to a continual *welding* together and tearing apart of the smaller parts of the surfaces as they come into intimate contact.

When the sliding surfaces are distinctly rough there is no regularity whatever in the friction. Surfaces which are comparatively smooth satisfy the law very closely, especially if they are of different materials and are not allowed to rest, for at starting the friction is always greater. Very true, clean surfaces of the same materials when brought into really close contact, can scarcely be made to slide at all. Glass will not slide on glass if very clean, provided the two faces have been well pressed together, so as to exclude the intervening layer of air. If laid together gently, however, so that an air cushion intervenes, the same surfaces will be found apparently almost devoid of friction. In the same way surfaces are *lubricated* by means of a *thick* layer of liquid, sliding with great freedom; but they may be made to cling almost as though cemented if they are separated by a sufficiently thin layer of the same lubricant. In the former case the friction is merely that which arises from the viscosity of the lubricant.

165. Experimental determination of the coefficient of friction. — To measure sliding friction, one of the two following methods is usually employed:

(1) *Method of the horizontal plane.* — Upon the bed *AB* (Fig. 85) are placed the two materials, the friction between which is to be determined. These should be in the form of blocks with plane faces. One of them, which should present as long a sliding surface as convenient, is fastened to the bed.

To the other a horizontal force is applied by means of a cord running over a pulley and sustaining weights at its free end. The sliding block, the mass of which must be known, is loaded with known weights. The normal force pressing the surfaces together is Mg, where M is the sum of the masses of block and load. The weights (m), which are applied to the cord, must suffice to maintain the sliding block, once started, in uniform motion. The force which they then exert upon the vertical portion of the cord (mg) is that necessary to overcome the friction of the moving wheel, and the friction of the surfaces to be tested.

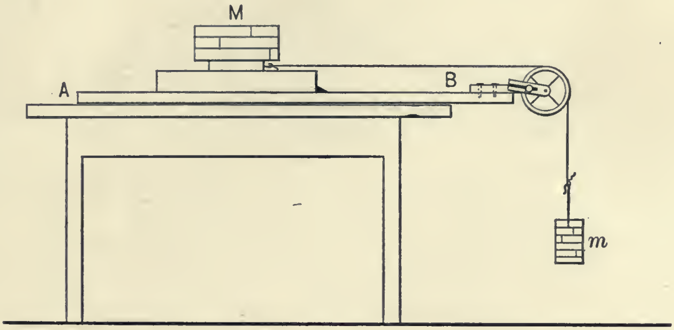

Fig. 85.

Assuming the former to be negligible, we have for the coefficient

$$\mu = \frac{m}{M}. \tag{112}$$

In point of fact, it is necessary to ascertain the correction for the friction of the wheel, since on account of it the horizontal pull is less than the vertical by two per cent or more.

By means of this apparatus, if the load M be varied, and likewise the areas of the surfaces in contact, Coulomb's law is readily verified.

(2) *Method of the inclined plane.*—An inclined plane AB (Fig. 86), the angle of which can be varied at will, serves as a

bed for the fixed block. Upon this the sliding block with its load is placed. The angle of the plane is varied until the block, once started, slides freely with a uniform motion. We then have the coefficient measured by the tangent of the angle of the plane (see equation 111).

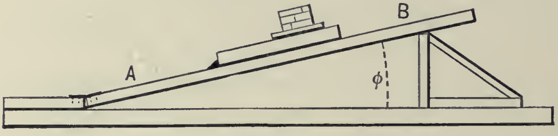

Fig. 86.

CHAPTER VII.

HYDROMECHANICS.

I. Fluids at Rest; Hydrostatics.

166. Pressure in a fluid. — A fluid is a substance which when at rest can sustain no stress other than hydrostatic pressure. When it is under hydrostatic pressure, any given plane of area a, exposed to the fluid, is acted upon by a force F such that

$$F = ap, \tag{113}$$

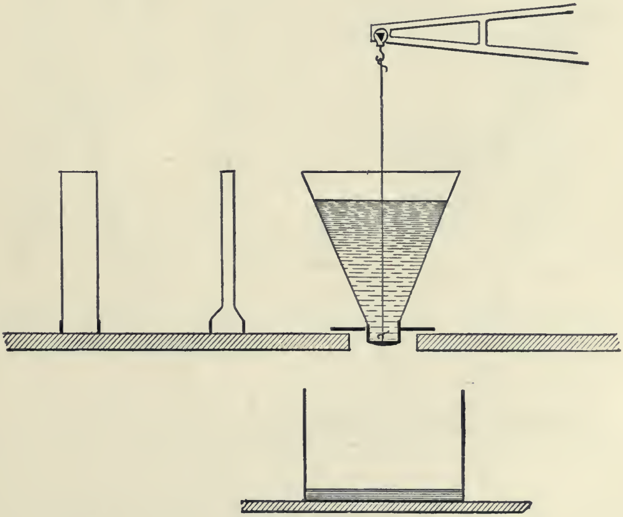

Fig. 87.

in which p is the value of the hydrostatic pressure. That the force F is normal to the area when the fluid is at rest (*Pascal's principle*) has already been shown (Art. 131). It has also been shown that in the case of fluids sensibly incomprehensible (Art. 148) a horizontal section at a distance x below the surface is acted upon by a force $axdg$, where a is the area, d the density of the fluid; and that the pressure at any point in this section, due to the weight of the liquid above it, is

$$p = xdg. \qquad (93\ bis)$$

This principle is usually verified experimentally by means of *Pascal's vases.* The apparatus consists of a series of glass vessels (Fig. 87) varying greatly in form and content, but all of the same vertical height, and all fitting the same movable base, which is held in place by suspension from one arm of a balance. When the pressure p upon the bottom of the vessel overbalances the sustaining weights in the scale pan at the other arm, the base is dislodged and the vessel emptied. The experiment consists in showing that whatever be the form of the vessel, this occurs for a given sustaining weight when the liquid reaches a certain depth.

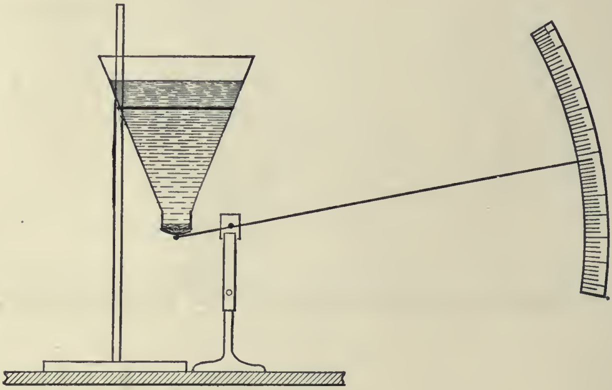

Fig. 88.

A better result may be obtained by indicating the pressure at the bottom of the vessel by means of the distension of an elastic membrane stretched across the opening in place of the movable base. This may be made to act upon a pointer which moves along a scale (Fig. 88).

167. The law of Boyle and Mariotte.—The relationship between the volume and pressure of a gas discovered by Boyle (1662), and independently and more completely established by Mariotte (1676), is expressed by the equation

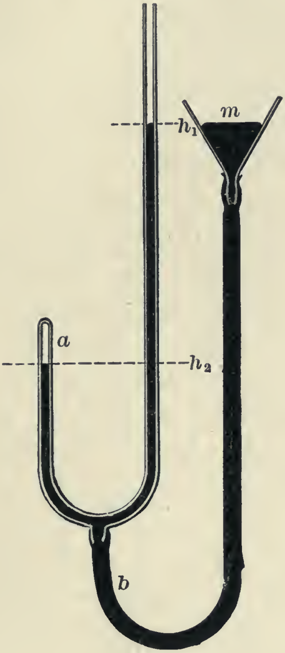

Fig. 89.

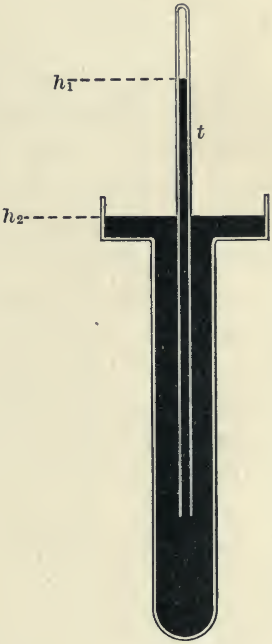

Fig. 90.

$$pv = C, \qquad (114)$$

in which p and v are pressure and volume, respectively, and C is a constant.

The following is the statement which corresponds to this equation; viz.:

In a true gas the volume is inversely proportional to the pressure to which the gas is subjected.

It is known as Mariotte's (or Boyle's) law. See Arts. 249, 256.

The experimental verification of Boyle's law is usually performed by means of the apparatus shown in Fig. 89. It consists of a U-shaped tube, the shorter arm of which is closed. Through an opening at b mercury from the reservoir m enters both arms of the tube. By raising and lowering m the height of the column ending at h_1 can be adjusted. Observations of the volume of the entrapped air at a, the pressure of which is always one atmosphere plus that due to the vertical column of mercury h_1h_2, is found to be that required by the law.

For the extension of the experiment to pressures less than one atmosphere the apparatus is frequently given the form shown in Fig. 90. It consists of a glass tube (t) closed at one end and inverted in a reservoir of mercury. The inverted tube likewise contains mercury, but a small amount of air is entrapped within its upper end. The volume of this enclosed air varies with the pressure, which is always one atmosphere less that due to the vertical column of mercury h_1h_2. The latter may be varied by raising and lowering the tube t, and the variations of volume thus produced may be observed.

168. Limitations of Boyle's law. — All gases, when subjected to sufficient pressure under temperature conditions which are described in Chapter IX are converted into liquids. As the conditions under which this change of state occurs are approached, the gas gradually takes on new properties. When compressed, its loss of volume is greater than would be expected from Mariotte's law, the divergence from inverse proportionality increasing as we near the point of condensation. Substances in this intermediate condition preparatory to liquefaction are called *vapors*.

Boyle's law applies only to *true gases*, to gases, that is to say, which are very far removed, as to their condition, from the boiling-point or point of liquefaction.

Many substances, such as chlorine (Cl), ammonia (NH_3), hydrogen disulphide (H_2S), sulphur dioxide (SO_2), nitrous oxide (N_2O), and carbon dioxide (CO_2), which assume gaseous form at ordinary temperatures, are readily liquefied by the application of pressures. Thus, at 0° C., these substances become liquid at the pressures indicated in the following table:

TABLE III.

PRESSURES NECESSARY TO CONDENSE CERTAIN VAPORS AT 0° C.

Sulphur dioxide (SO_2) . .	1.51 atmos.	Hydrogen disulphide (H_2S)	10.80 atmos.
Chlorine (Cl)	3.66 "	Carbon dioxide (CO_2) . .	35.40 "
Ammonia (NH_3)	4.19 "	Nitrous oxide (N_2O) . . .	36.08 "

To show that vapors are more compressible than true gases, two closed glass tubes, t_1, t_2, with strong walls (Fig. 91), are filled respectively with a vapor, such as ammonia, and nitrogen (or air). The lower ends of these tubes are submerged in a closed reservoir of mercury, the walls of which are of massive cast iron, and pressure is applied by means of a compression pump. The mercury rises in both tubes as the gaseous contents are compressed, but it rises faster in the tube containing the vapor than in that the contents of which obey Mariotte's law. Finally, when the necessary pressure has been reached, the vapor condenses.

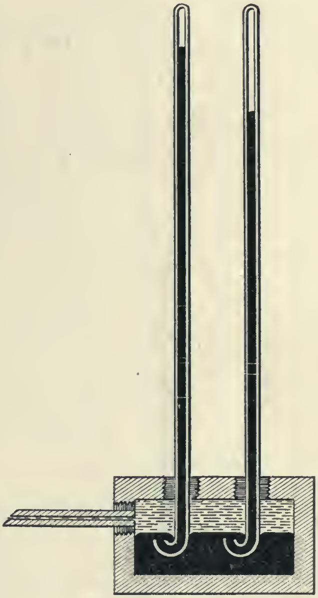

Fig. 91.

169. Pressure in a compressible fluid such as the atmosphere. — Let P be the atmospheric pressure at a given point, and d the density of the air at that point. Let $P+\Delta p$ be the pressure at a point distant Δx below the given point; then from equation (94) we have $\Delta p = d \cdot g \cdot \Delta x$. From Boyle's law we have $d = kp$, *i.e.* the density of the air is *proportional* to the pressure, temperature being supposed constant. Substituting kp for d in $\Delta p=d\cdot g\cdot\Delta x$, we have $\frac{\Delta p}{p}=kg\Delta x$. Integrating, we have $\log_e p=kgx+$ constant. When $x=0$, $p=P$; therefore, the constant of integration is equal to $\log_e P$, so that

$$\log_e p=kgx+\log_e P, \tag{115}$$

or

$$p=Pe^{kgx}, \tag{116}$$

where p is the atmospheric pressure at a point distant x below the given point at which the pressure is P.

Remark. — If the atmospheric pressure p and P at each of two points be observed, and the constant k and acceleration of gravity g be known, the vertical distance x between the two points can be calculated from (116).

170. Measurement of pressure by means of the barometer. — Consider a tube T (Fig. 92) filled with mercury and inverted in a cistern of mercury C, the tube being of such length that an empty space V is left in which the pressure is zero. Let the pressure p to be measured act at the surface of the mercury in the cistern. The pressure (p) inside of the tube at the same level (see Art. 166 and equation 93) will be

$$p=xdg, \tag{93 bis}$$

so that p may be calculated when x has been observed and d and g are known. If d and g have always the same value, x itself may be taken as the *measure* of the pressure. The unit pressure in this case is equal to the pressure at unit distance beneath the surface of mercury under standard conditions as regards d and g.

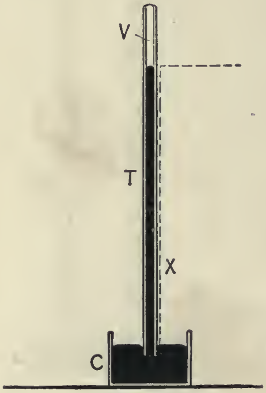

Fig. 92.

171. Measurement of pressure by means of manometers. — The barometer is used for the most part to indicate the slight changes in atmospheric pressure with which the meteorologist has to do. The corresponding instrument for the measurement of artificial pressure is called a *manometer.*

Of the numerous forms of manometer or pressure gauge, the following are among the most important.

(*a*) **Open tube manometer.** — The essential feature is a U-shaped tube (Fig. 93) both arms of which are open. This is partly filled with a liquid, generally mercury. One arm is connected with the vessel (*a*) the pressure of the contents of which is to be measured. The difference in the height of the columns of liquid in the two arms of the instrument measures the difference in pressure.

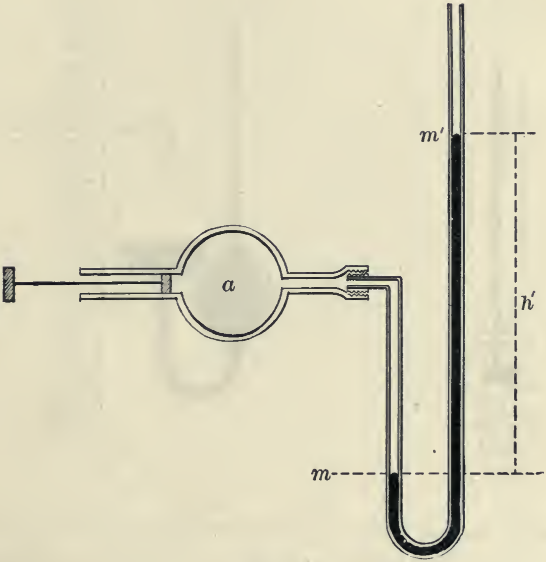

Fig. 93.

If the atmospheric pressure (h) be expressed in centimeters of mercury, and the vertical height h' between the crests of the mercury columns (m and m', Fig. 93) be also given in centimeters, the pressure at a is

$$p = h + h'.$$

(*b*) **Closed-tube manometers.** — For the measurement of considerable pressures, the length of tube required in the construction of an open-air manometer renders that form of apparatus inconvenient and in extreme cases impracticable. In such cases the open arm is shortened and sealed, and the shrinkage in the volume of the enclosed air is used to indicate the pressure.

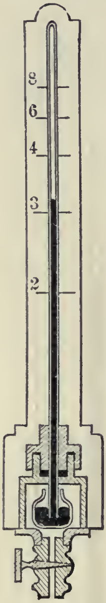

Fig. 94.

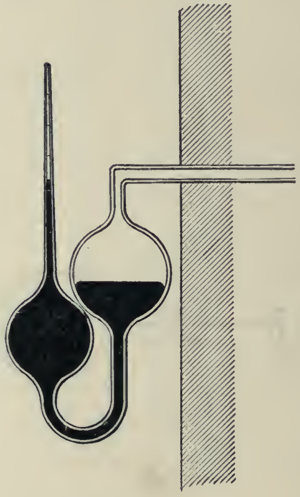

Fig. 95.

The tube of such manometers is sometimes cylindrical as in Fig. 94, in which case the distances between successive divisions of the scale form members of the diminishing series

$$1 : \tfrac{1}{3} : \tfrac{1}{6} : \tfrac{1}{10} : \tfrac{1}{15} : \tfrac{1}{21}, \text{ etc.}$$

Sometimes the closed tube manometer is given the conical form shown in Fig. 95, a form in which the scale is more nearly uniform.

(*c*) **Manometers for very high pressures.**—When the pressures to be measured are large, the volume of air within the closed-tube manometer becomes so small as to make accurate observations of its changes difficult; recourse is then had to one of the numerous forms of mechanical pressure gauge. Figure 96 shows a well-known form, that of Ducomet, in which the pressure acting upon the interior of the flexible wall of a capsule (*C*) against which the double spring *AB* acts from without, gives a greatly magnified movement to a pointer. For the determination of excessively high pressures, Cailletet (1880), in some of his researches upon the liquefaction of the permanent gases, made use of the compressibility of glass. His manometer (Fig. 97) consisted of a bulb of glass containing mercury. This was enclosed in a massive cylinder of steel which was connected with the reservoir, the pressure within which was to be measured by means of a copper tube of heavy walls and small bore. The compression of the glass caused the enclosed mercury to rise, indicating pressures upon an attached scale.*

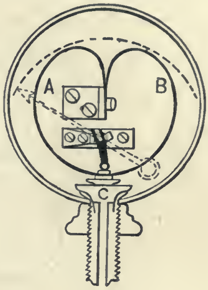

Fig. 96.

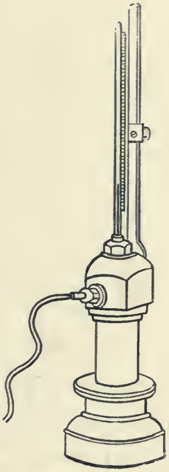

Fig. 97.

(*d*) **The McLeod gauge.** — It is sometimes necessary to measure extremely low pressures, for which purpose neither the barometer nor the open-tube manometer, nor, indeed, the closed-tube manometer in its usual form is adapted. The apparatus commonly employed is called (from its inventor) the McLeod gauge. It consists of a glass bulb (Fig. 98) of known capacity, carrying a closed capillary tube (*t*), the contents of which for each centimeter of its length is likewise known. This bulb is connected

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* Cailletet, Annales de Chimie et de Physique (5), 19, p. 388; 1880.

with the vessel *V*, the internal pressure of which is to be ascertained through the tube *p*. It is separated from the outer atmosphere by the column of mercury *m*, the height of which can be varied by raising or lowering the reservoir *M*. The gas within the bulb being in connection with that of the vessel *V*, is at the same pressure. By raising *M*, however, it may be isolated from *V* by the intervention of the mercury. The gas thus isolated is compressed by causing the rising mercury to flow into the bulb, until, finally, it is brought to a pressure of one atmosphere. The volume which it then occupies will be a small portion of the contents of the capillary tube *t*. From the relative volumes the initial pressure is readily computed by Mariotte's law (Art. 167).

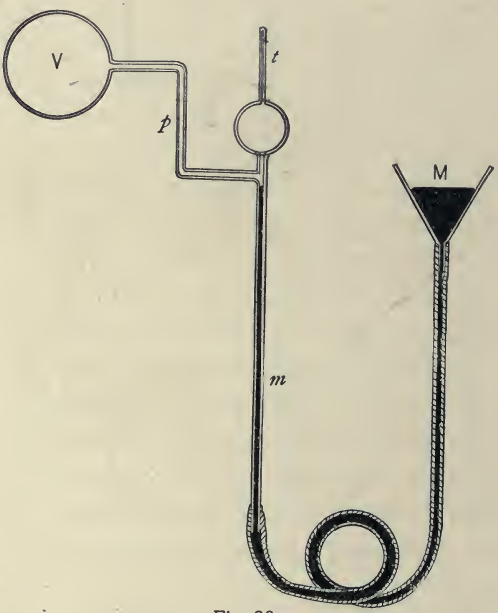

Fig. 98.

172. Archimedes' principle. — The total force with which a fluid acts upon a submerged body is equal to the weight of the displaced fluid, or, in other words, to the weight of a volume of the fluid which equals the volume of the body. It is a vertical force, and its point of application is at the point which would be the center of mass of the submerged body, were the latter homogeneous.

Proof. — Consider a *portion* of the fluid of the same size and shape as the submerged body. This portion of fluid is stationary, so that the total force with which the surrounding fluid acts upon it must have the same point of application as the force due to gravity, and must be equal and opposite thereto. The force due to gravity is of course the weight of the portion of fluid, and its point of application is the center of mass thereof; so that the proposition is proven.

Corollary. — If the body is but partly submerged, the total force with which the fluid acts upon it has its point of application at the center of mass of that portion of the body (considered homogeneous) which lies below the plane of the fluid surface, and is equal and opposite to the weight of a portion of the fluid of the same size and shape.

Remark. — The total force with which a fluid acts upon a body wholly or partly submerged is called the **buoyant force** of the fluid, and its point of application is called the **center of buoyancy.**

173. Archimedes' principle in the case of gases. — When a body is submerged in a gas, such as the earth's atmosphere, which is the case with most substances when they are being weighed, it is acted upon by the buoyant force of the fluid, in accordance with the principle of Archimedes. The effect, though slight on account of the small density of the surrounding fluid, is by no means inappreciable, and it has to be considered in the accurate weighing of bulky objects. It is commonly demonstrated qualitatively, by means of an instrument called the baroscope (Fig. 99). This consists of a hollow sphere of glass or metal, at one arm of a small balance, the same being counterpoised by a leaden weight. It is easy to make the adjustment of the balance such that the globe is slightly heavier in air, but when tested in vacuo, by placing the baroscope under the bell jar of an air pump and exhausting the air, it is in excess of the leaden weight. The loss suffered by each, in air, will obviously be that due to the buoyant force of the displaced air; a force much larger in the case of the hollow globe than in that of the compact counterpoising weight.

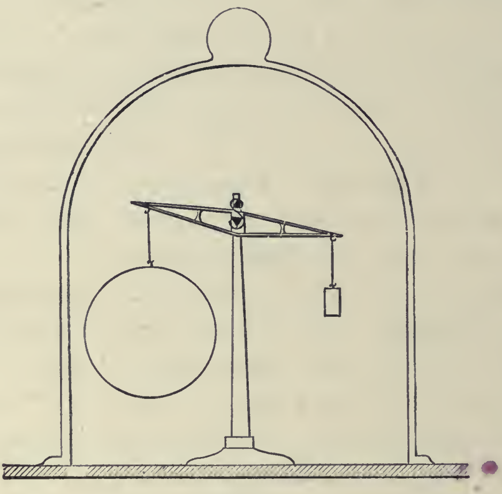

Fig. 99.

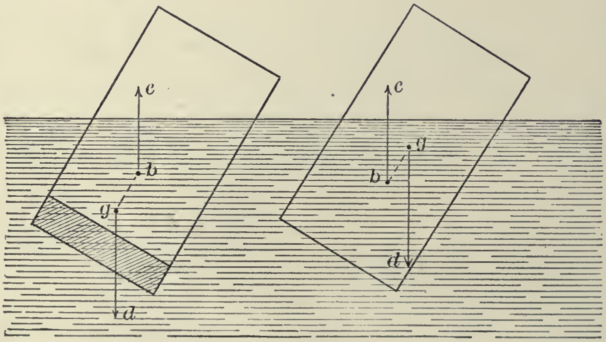

Fig. 100.

174. Floating bodies. — A body floating upon a fluid displaces its weight of the fluid. Its power to do so (or in other words the fact that its mean density is less than that of the fluid) determines whether it will float freely or will be entirely submerged.

When the center of mass and the center of buoyancy are in the same vertical line, the body is in equilibrium. That this is the condition of equilibrium is evident since in general a floating body is acted upon by a couple *gd* ⋯ *be* (Fig. 100), tending whenever the line *bg* is not vertical to render it so.

175. The metacenter. — The question of the stability or instability of the equilibrium of a floating body is somewhat complicated by the fact that the center of buoyancy shifts with the position of the body. When equilibrium exists, the line *gb* is called the axis. When the body is infinitesimally inclined, the center of buoyancy is shifted to b'. A vertical through this new position will cut *bg* in *m*, which point is called the *metacenter*. When the metacenter is above the center of mass as in Fig. 101, the equilibrium is stable; when below, it is unstable.

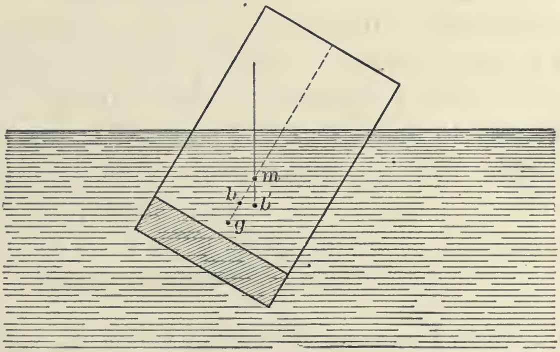

Fig. 101.

PROPERTIES OF THE SURFACE FILM OF LIQUIDS — CAPILLARITY.

176. Sphere of molecular action. — It is probable that *cohesion* and *adhesion*, those forces, namely, which cause the similar and dissimilar particles of which bodies are made up to cling together, are of the same nature as the gravitational attraction of particles at sensible distances.

It can be shown, however, that the attraction must increase more rapidly than the inverse square of the distance at very small distances, in order to account for the extremely great forces of cohesion and adhesion, and the sudden cessation of these forces at (still very small) sensible distances.

Since molecular* attraction increases much more rapidly than the inverse square of the distance, the action of the more remote particles of a body upon a given particle is negligible compared with the action of the particles in the immediate neighborhood. A small sphere, with a radius perhaps not greater than a few millionths of a centimeter, which includes all the particles having appreciable action upon the particle at its center, is called the **sphere of molecular action**.

177. Increase of pressure in passing across the surface of a liquid. — The surface layer of a liquid, of thickness equal to the radius of the sphere of molecular action, is called the **surface film.**

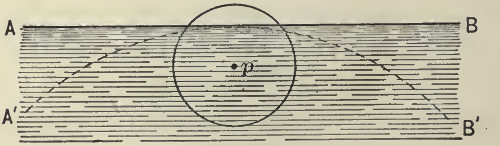

Fig. 102.

Let *AB* (Fig. 102) be the plane surface of a liquid. Consider the molecular forces acting upon the fluid particle *p*. Describe

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* The word molecular is to be here understood as referring merely to the smaller parts of a body.

the sphere of molecular action with its center at p. If this sphere lies wholly within the liquid, that is, if p lies inside of the surface film, the forces acting on p are the same in all directions. If p lies *in* the surface film as shown, there will be on the whole a downward force on p, because of the greater number of attracting particles below than above p. Such forces upon the particles of a liquid near its surface make the hydrostatic pressure just inside the surface film greater than just outside the liquid. This increase of pressure across the plane surface of a liquid is called the **molecular pressure** of the liquid.

If the surface of the liquid is convex, as $A'B'$ (Fig. 102), there will be for each position of p a greater excess of attracting particles below than for a plane surface, so that the increment of pressure across a convex liquid surface is greater than it is across a plane surface. Similarly, the increment of pressure across a concave surface is less than it is across a plane surface.

178. Surface tension and surface energy.

(*a*) A surface film, or membrane, is said to be under tension when the portions of the membrane on the two sides of any line pull on each other. The **force per unit length of line** is the measure of the tension.

(*b*) To increase the area of a surface, or membrane, under tension, work must be done. The **work done per unit increment of area** is called the potential energy of the surface, or simply the **surface energy**.

179. Proposition. — The tension of a surface is equal to its surface energy.

Proof. — Consider a rectangular portion, $ABCD$ (Fig. 103) of a membrane under tension T. The force F pulling on the side AD to maintain the tension is $F = Tx$. Let the membrane be stretched by an amount Δy in the direction of F. The work done by the force F is $F\Delta y = Tx\Delta y$, which, divided by the increment of area $x\Delta y$, gives T as the work done per unit increment of area.

Corollary. — Any surface which requires work to be done to increase its area is under a tension equal to its surface energy.

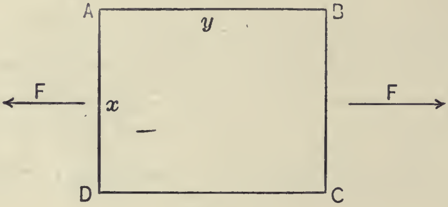


Fig. 103.

180. Surface energy and surface tension of a liquid. — From Art. 177 it is evident that work must be done to carry a particle of a fluid from its interior into the surface film. If the total area of the fluid surface remains unchanged some other particle of the fluid will pass from the surface film into the interior and, on the whole, no work will be done. If, however, the movement of interior particles to the surface of a liquid is a result of an increase in the extent of the surface, an amount of work proportional to the increase of surface will be expended. A liquid has therefore a definite **surface energy**; therefore the surface film of a liquid is under a definite **tension.** Compare Arts. 178 and 179.

181. Angles of contact; case of three fluids. — Let a (Fig. 104) be a line perpendicular to the paper, at which three fluid

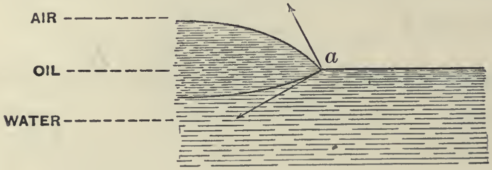
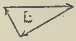


Fig. 104.



surfaces meet. The condition of equilibrium requires the tangents of the three surfaces at a to be parallel to the sides of a triangle B, these sides being proportional to the tension of the respective surfaces. The angles between the surfaces at a can thus be calculated. They are always the same for the given fluids, and are called the **angles of contact**.

182. Angles of contact; case of two fluids and a solid. — Not only is a surface, separating two fluids, under tension, but so also is a surface separating a solid and a fluid. In this case, however, the surface being rigid, the experimental evidence is less complete. Let a (Fig. 105) be the line of contact, with glass, of the surface separating oil and water. The condition of equilibrium requires the resolved part parallel to AB, of the tension S, to be equal to the difference of the tensions o and w, so that the angle θ is thereby determined.

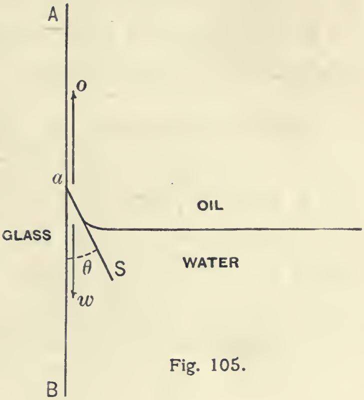


Fig. 105.

Remark. — Very slight impurities in a liquid, or the least contamination of its surface, alters the surface tension greatly.

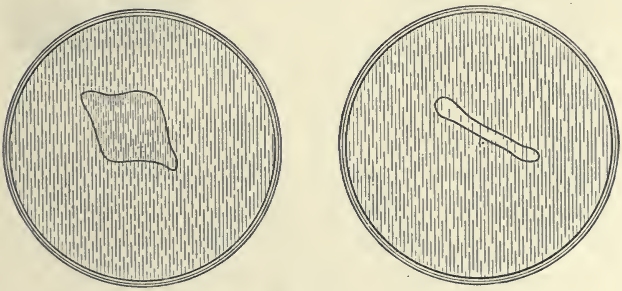


Fig. 106.

This fact may be demonstrated by means of the following very simple but effective experiment.

Upon a vessel of water, an ordinary thin rubber band is floated. This divides the surface into two regions, external and internal.

If the internal region be merely touched with a glass rod or splinter of wood which has previously been dipped in some other liquid such as alcohol or oil, its surface tension is diminished. The tension of the outer region, as yet unchanged, instantly draws the hitherto nearly closed loop into an approach to circular form (Fig. 106). If the film of the external region be then weakened by a similar application, the band relapses again into its normal shape.*

183. Capillary elevation and depression. — Let *AB* (Fig. 107) be a broad vessel containing water, in which a capillary tube, *T*, of glass is placed. The surface of the water at *a* is sensibly *plane;* in the capillary tube, however, the surface of the water is concave as shown. The pressure of the surrounding air is everywhere the same, so that the pressure just inside the plane surface at *a* must be greater than the pressure just inside the concave surface in the tube. [See Art. 177.] This *difference of pressure* causes the water to rise in the tube, until it is balanced by the difference in level *h*.

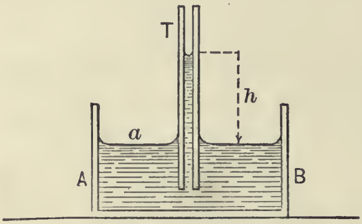


Fig. 107.

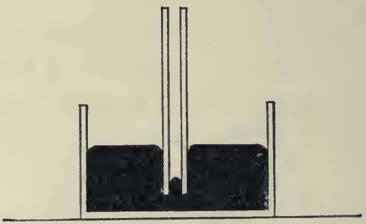


Fig. 108.

If the vessel, described above, contains mercury instead of water, the surface within the capillary tube will be convex. Consequently the pressure just beneath this surface will be greater than within the surface in the surrounding vessel, and the mercury in the tube will be depressed as shown in Fig. 108.

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* See Ernest Nichols, Physical Review, Vol. I., p. 299.

184. The law of diameters (Jurin's law). — The elevation or depression of a liquid in a capillary tube is inversely proportional to the diameter of the tube. This is sometimes called *Jurin's law*, but being a result of the above theory of molecular attraction, it is properly an experimental verification thereof.

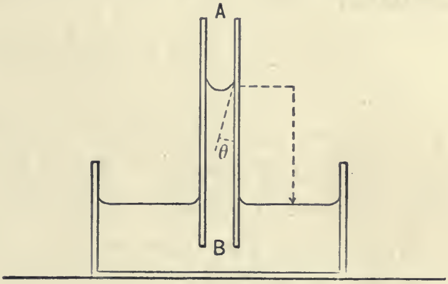

Fig. 109.

Jurin's relation between size of capillary tubes and the amount of elevation or depression therein is derived from the foregoing principles as follows:

Let Fig. 109 represent a capillary tube, bore of radius r, plunged in a broad vessel of liquid. Then $\pi r^2 hdg$ is the weight of the elevated column h of liquid, $\pi r^2 h$ being its volume, d its density, and g the intensity of gravity. The force which sustains this weight is the vertical component, $2\pi rT\cos\theta$, of the force due to the tension of the liquid surface at the walls of the tube, $2\pi r$ being the circumference of the bore of the tube, T the surface tension, and θ the angle of contact. Therefore

$$\pi r^2 hdg = 2\pi rT\cos\theta,$$

whence

$$h = \frac{2T\cos\theta}{dgr}. \tag{117}$$

For water in glass tubes $\theta = 0$, so that for this case (117) becomes

$$h = \frac{2T}{dgr}. \tag{118}$$

This equation enables the calculation of T when h, d, g, and r are known.

185. Equation of capillarity. — The difference in pressure on the two sides of a liquid film, due to the tension of the film, depends upon the tension and upon the curvature of the film. The equation expressing this relation is called the **equation of capillarity.** It is

$$p = T\left(\frac{1}{R'} + \frac{1}{R''}\right), \tag{119}$$

in which p is the difference in pressure on the two sides of a film, due to the tension T of the film, and R' and R'' are the principal radii of curvature of the film at the point.

Proof.—(*a*) The tension of a film is equivalent to two mutually perpendicular equal pulls. The effect of each pull may in any case be considered by itself, the total effect being the sum of the effects due to each.

(*b*) Consider a small portion ΔS of any curved surface in the neighborhood of a chosen point on the surface. Draw a tangent plane to the surface at the point. A plane normal to this tangent plane intersects ΔS in a curve which is sensibly an arc of a circle of radius r. There is one position of this normal plane for which r has a maximum value R', and another position at right angles thereto for which r has a minimum value R''. These two radii, R' and R'', are called the *principal curvatures* of the surface at the point. The effects of each of these two curvatures of a surface at a point may in any case be considered separately. When considering the effect of one curvature the surface may be thought of as cylindrical.

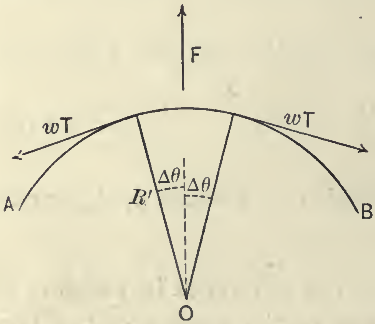

Fig. 110.

(*c*) **Pressure inside a cylindrical film.**—Let AB (Fig. 110) represent a strip, of width w, of a cylindrical film of radius R'. Consider the portion of this strip which subtends an angle $2\,\Delta\theta$ from the axis O of the cylinder. This portion is of length $2\,\Delta\theta\cdot R'$ and of width w; therefore the *force due to inter-*

nal pressure p pushing out on this portion is $pwR' \, 2\Delta\theta$. The forces wT, due to the tension T of the film, pulling this portion of the film on either side, are inclined at the angle $\Delta\theta$ to the horizontal, so that the downward component of each is $wT \cdot \Delta\theta$. Therefore *the total downward force acting upon the portion of the film under consideration is $2wT \cdot \Delta\theta$.*

It follows that

$$2wT \cdot \Delta\theta = pwR' \, 2\Delta\theta \quad \text{or} \quad p = \frac{T}{R'}. \tag{120}$$

(d) In case of a film having double curvature, with principal radii R' and R'', the internal pressure contributed by each is $\frac{T}{R'}$ and $\frac{T}{R''}$ respectively, so that the total internal pressure is

$$\frac{T}{R'} + \frac{T}{R''}. \qquad \text{Q.E.D.}$$

186. Examples of the application of the equation of capillarity.

(a) **Liquid uninfluenced by gravity.** — In such a liquid the pressure is the same throughout, so that from equation (119) $\frac{1}{R'} + \frac{1}{R''}$ must have the same value at all points of the surface of such a liquid. The only finite surface which satisfies this condition is the sphere. Therefore the surface of a liquid uninfluenced by gravity must be spherical.

Gravity produces only imperceptible variation of pressure at different points in a very small globule of liquid so that such globules always assume a spherical form.

(b) **Soap bubble.** — The difference in pressure inside and out of a soap bubble is sensibly of one value at every point of its surface, so that $\frac{1}{R'} + \frac{1}{R''}$ is constant, and the bubble is spherical.

(c) **Soap films on wire frames** (Plateau). — In this case the pressure is the same on the two sides of the film, so that

$$\frac{1}{R'} + \frac{1}{R''} = 0.$$

If the wires of the frame lie in a plane, the film itself will be plane; otherwise R' and R'' are equal and opposite at each point of the film.

FLUIDS IN MOTION; HYDRAULICS.

187. Fluid friction; viscosity. — The action of a shearing stress upon a portion of fluid is to continually lengthen it in the direction of the *pull*, and shorten it in the direction of the *push*. That is, ϕ being the angular distortion of a portion of the fluid, the action of a shearing stress is to cause ϕ to change at a definite *rate*, $\frac{d\phi}{dt}$.

Stokes' law. — Stokes found that the rate, $\frac{d\phi}{dt}$, at which the angular distortion of a portion of fluid changes is approximately proportional to the shearing stress. The constant ratio, shearing stress divided by $\frac{d\phi}{dt}$, for a given fluid is called its **coefficient of viscosity.** We may write, therefore,

$$\eta = \frac{S}{\frac{d\phi}{dt}}, \tag{121}$$

in which η is the coefficient of viscosity of a fluid, and $\frac{d\phi}{dt}$ is the rate at which it is distorted by a shearing stress S.

188. Viscous fluid between parallel plates; lubrication. — Let $A'B'$ (Fig. 111) be a plane plate, and AB another, parallel thereto, distant x therefrom, and moving at velocity v, as indicated by the arrow. Let the space between the plates be filled with a viscous fluid. Consider a portion CD of the fluid, which at a given instant is cubical. After the lapse of time Δt, the upper plate will have moved a distance $v\Delta t$, and this cubical portion of fluid will be distorted as indicated by the dotted rhombus, of which the angle of distortion is $\Delta\phi = \frac{v\Delta t}{x}$. Therefore

$$\frac{d\phi}{dt} = \frac{v}{x}. \tag{122}$$

The stress in the liquid, aside from hydrostatic pressure, is a shearing stress of which the two pulls are $+S$ and $-S$ in the directions shown. Substituting $\frac{d\phi}{dt}=\frac{v}{x}$ in (121), we have

$$S=\frac{\eta v}{x}. \tag{123}$$

The stress on all horizontal (also on all vertical) sections in the fluid is purely tangential and equal to S. Therefore the plates are acted upon by dragging forces $F=Sa$, in which a is the area of the plate. Substituting the value of S from (123), we have

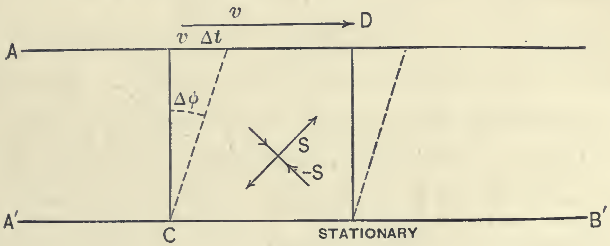

Fig. 111.

$$F=\frac{a\eta v}{x}, \tag{124}$$

in which F is the force required to slide a plate of area a at velocity v over another with a layer between, of thickness x, and of which the coefficient of viscosity is η. The coefficient of viscosity of a fluid is sometimes defined directly from this equation, instead of from equation (121). Equation (124) can be easily transformed to apply to a flat circular disk rotating near a flat plate with viscous fluid between.

189. Flow of viscous fluid through a long tube. — Let ρ be the radius of the bore of the tube, l its length, p the pressure forcing the fluid through, and v the velocity of the fluid at a distance r from the axis of the tube. Consider a cylindrical portion of the fluid of radius r coaxial with the tube. The surface of this cylindrical portion moves as a solid *rod* through the tube at velocity v. Similarly the cylindrical surface of radius $r + \Delta r$ moves through the tube as a hollow *shell* at velocity $v + \Delta v$. The fluid layer between this *rod* and *shell* is under conditions similar to the fluid between the plates, as discussed in the previous article. Therefore, writing Δv for v in equation (122), and Δr for x, we have $\frac{d\phi}{dt} = \frac{dv}{dr}$. Substituting this value of $\frac{d\phi}{dt}$ in (121), we have $S = \eta \frac{dv}{dr}$. This stress is tangential over the whole surface, $2\pi rl$, of the *rod*, so that the dragging force on the *rod* is $2\pi rlS$ or $2\pi rl\eta\frac{dv}{dr}$. This dragging force is continually overcome by the force $\pi r^2 p$ pushing on the end of the *rod*, and is equal thereto. We have, therefore, $\pi r^2 p = 2\pi rl\eta \frac{dv}{dr}$, or $\frac{dv}{dr} = \frac{p}{2l\eta}r$, or $v = \frac{pr^2}{4l\eta} + constant$. When $r = \rho$, $v = 0$, so that the constant of integration is equal to $-\frac{p\rho^2}{4l\eta}$. It follows, therefore, that

$$v = \frac{pr^2}{4l\eta} - \frac{p\rho^2}{4l\eta}. \qquad (125)$$

The velocity at each part of the tube is thus determined. To find the volume V of fluid discharged in time τ, consider a section of the tube (Fig. 112). The velocity over all the area, $2\pi r\Delta r$, of the dotted annulus, is v, so that the volume ΔV, flowing across this annulus in time τ, is $\Delta V = 2\pi r\Delta r \cdot v \cdot \tau$. Substituting v from (125), we have

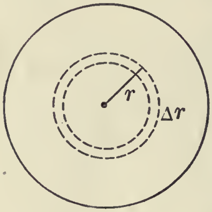

Fig. 112.

$$dV = \frac{\pi p\tau}{2l\eta} r^3 dr - \frac{\pi p\rho^2\tau}{2l\eta} rdr,$$

or

$$V = \frac{\pi p\tau}{2l\eta}\int_0^\rho r^3 dr - \frac{\pi p\rho^2\tau}{2l\eta}\int_0^\rho rdr.$$

Therefore

$$V = \frac{\pi p\rho^4\tau}{8l\eta}. \qquad (126)$$

This equation enables the calculation of η when V, p, ρ, τ, and l have been determined.

MOTION OF INCOMPRESSIBLE, FRICTIONLESS FLUIDS.

190. Energy of a liquid.

(*a*) **Potential energy.** *Proposition.* — The work W, required per unit volume to introduce incompressible fluid into a region under pressure p, is

$$W = p. \tag{127}$$

Proof. — Let CC (Fig. 113) be a cylinder provided with a piston of area a, and communicating with a vessel under pressure p. The force F necessary to move the piston is ap (Eq. 113). The work done in pushing the piston a distance Δx is $ap\Delta x$, from (55); but $a\Delta x$ is the volume of liquid forced into the vessel by the movement, so that the work done per unit volume is p.

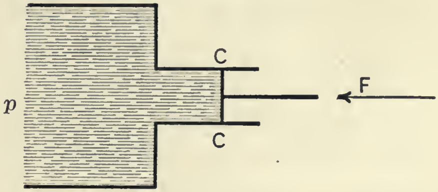

Fig. 113.

*Corollary.** — The **potential energy per unit volume** of an incompressible fluid under pressure p is equal to p, for that amount of work must be done to introduce unit volume of the fluid into a region under pressure p.

(*b*) **Kinetic energy.** — The kinetic energy per unit volume at a point in a moving fluid is

$$W = \tfrac{1}{2} dv^2, \tag{128}$$

in which d is the density of the fluid, and v is its velocity at the point.

Proof. — Consider a small volume ΔV of the fluid. Its mass is $\Delta V \cdot d$, whence, from equation (59), its kinetic energy is

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* This statement, although true in the case in which it is to be used below, requires for generality a number of limiting conditions to be stated. The general aspect of such limitation is as follows: Work *done* is always expressed as the product of two factors, one of which is a generalized *co-ordinate* and the other is a generalized *force*. For examples, see equations (55), (57), (106), (107), (109), etc. The *doing* of this work always depends upon the variation of the *generalized co-ordinate*. Thus an incompressible fluid may be brought from zero pressure to any given pressure without the expenditure of *any* work by making the *pressure* the variable.

$\frac{1}{2}\Delta V \cdot dv^2$. This, divided by ΔV, gives $\frac{1}{2}dv^2$ as the kinetic energy per unit volume.

191. Application of the principle of the conservation of energy.

(*a*) A fluid having a large coefficient of viscosity is said to be **viscous.** Glycerine, syrup, and the like are viscous.

(*b*) A fluid having a small coefficient of viscosity is said to be **mobile.** Water, ether, and the like are mobile.

(*c*) A fluid of which the coefficient of viscosity is zero is said to be **frictionless.** There is no such fluid, but mobile fluids flowing through short pipes and orifices may be treated as though they were frictionless, without great error.

(*d*) An incompressible fluid is one which does not change its volume under pressure. Ordinary liquids may be treated as though they were incompressible, without great error.

(*e*) Generally speaking, energy is dissipated during the motion of a viscous fluid.

(*f*) No energy is dissipated in the motion of a frictionless fluid, **therefore the sum of the potential energy and kinetic energy of such a fluid must remain constant during the motion.**

192. Efflux.—Let V (Fig. 114) be a vessel containing a frictionless liquid of density d, under pressure p. Let oo be an orifice from which the liquid issues in a jet, the velocity of the particles of liquid in the jet being v and the outside pressure being p_1. At a point in the vessel far enough removed from oo that the velocity may be inappreciable, the energy (potential) per unit volume is p. In the jet the total energy per unit volume of the liquid is $p_1 + \frac{1}{2}dv^2$. From Art. 191 we have $p = p_1 + \frac{1}{2}dv^2$;

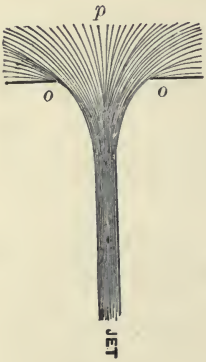

Fig. 114.

whence $$v = \sqrt{\frac{2(p - p_1)}{d}}. \qquad (129)$$

This equation is not strictly applicable to ordinary viscous compressible fluids. The effect of viscosity is to lessen v, and the effect of compressibility is to increase v. For ordinary liquids the effect of viscosity is the greater; therefore (129) generally gives too large values for v.

193. Vena contracta. — It is to be particularly noted that the discharge from the orifice, per unit time, is *not* va, a being the area of the orifice, inasmuch as v is not perpendicular to the plane of the orifice at each point. The lines of flow are somewhat as shown in Fig. 114. On account of this obliquity of the lines of flow in the orifice the jet contracts slightly after leaving the orifice. In this contracted vein (**vena contracta**) the velocity is sensibly uniform, in magnitude and direction, over the whole section of the jet; and the discharge per unit time is equal to the product of v into the sectional area of the jet at this narrow part.

Remark. — If the pressure in the vessel V (Fig. 114) is due to gravity, then $p - p_1 = xdg$, x being the distance of the orifice beneath the surface of the liquid in the vessel and g the intensity of gravity. Substituting this value of $p - p_1$ in (129), we have

$$v = \sqrt{2gx}. \tag{130}$$

From equations (23) and (24) this velocity is seen to be that which a body would gain in falling freely through the distance x (Torricelli's Theorem).

194. Diminution of pressure in a throat. — Let V (Fig. 115) be a vessel containing liquid of density d, at pressure p, having a tubular vent abc contracted at b. Let b and c be the sectional areas of the tube, v_1 and v_2 the fluid velocities, and p_1 and p_2 the pressures, at b and c respectively. From Art. 191 we have

$$p = p_1 + \tfrac{1}{2}dv_1^2 \tag{i}$$

$$p = p_2 + \tfrac{1}{2}dv_2^2. \tag{ii}$$

Further, bv_1 and cv_2 are each equal to the discharge per unit time, so that

$$bv_1 = cv_2. \tag{iii}$$

These equations (i), (ii), and (iii) completely determine v_1, v_2, and p_1 when p, d, p_2 and the ratio $b:c$ are given. The following two points are of particular interest.

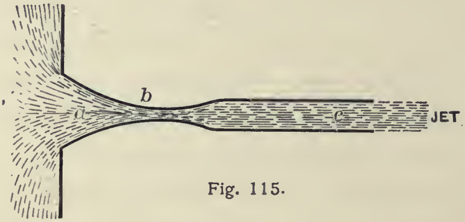

Fig. 115.

(a) Solving ii for v_2, we have

$$v_2 = \sqrt{\frac{2(p-p_2)}{d}},$$

which is identical to equation (129). **The efflux,** according to the above equation, *is unaffected by the narrow throat*, since the velocity in the broad tube at c is the same as if that tube were attached directly to the vessel.

(b) Solving for p_1, we have

$$p_1 = p - \frac{c^2}{b^2}(p - p_2). \tag{131}$$

Now p, c, b, and p_2 being entirely independent of each other, except only that $p-p_2$ is understood to be positive and $c>b$, it is evident that p_1 may be very small or even **negative.** This diminution of pressure in a throat is the explanation of the action of injector pumps and of some paradoxical toys.

Remark. — The following conditions* are assumed in the above discussion, in addition to the assumed frictionless character and incompressibility of the liquid, viz.:

* These two conditions are necessarily satisfied in a perfectly frictionless fluid, but not necessarily even **approximately** satisfied in a mobile fluid.

(*a*) That the wall widens gradually on the two sides of the throat so that no sharp corners may produce eddies.

(*b*) That the pressure p_1 in the throat does not attain to so great a negative value as to break the liquid into spray as it emerges into the widening portion of the tube.

195. Gauging water.

(*a*) **Orifice meters.** — In these meters the velocity of efflux of water from an orifice of known area is determined by the help of equation (130), x having been determined, and the discharge is then calculated from the area of the orifice and the velocity of efflux. The corrections required on account of the contracted vein are previously determined experimentally for the given shape of orifice.

(*b*) **Displacement meters.** — In these meters the flow is measured by counting the strokes of a piston in a cylinder so arranged that the cylinderful of water passes at each stroke.

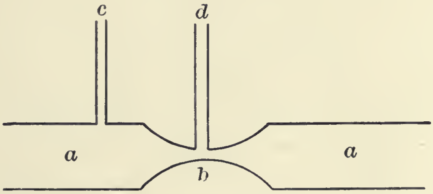

Fig. 116.

(*c*) **The Venturi meter.** — This meter depends upon the diminution of pressure in a throat, as follows :

Let *aba* (Fig. 116) be the pipe through which the water to be measured flows. Let v_1 and v_2, and p_1 and p_2, be the velocities and pressures at *a* and *b* respectively, and let *a* and *b* be the sectional area of the pipe at *a* and *b* respectively. From Art. 191 we have

$$p_1+\tfrac{1}{2}dv_1^2=p_2+\tfrac{1}{2}dv_2^2, \qquad \text{(i)}$$

also

$$av_1=bv_2, \qquad \text{(ii)}$$

av_1 and bv_2 being each the flow per unit time. Solving these two equations for av_1, we have

$$av_1 = \sqrt{\frac{2\,a^2b^2}{d(a^2 - b^2)}} \cdot \ (p_1 - p_2)^{\frac{1}{2}}. \qquad \text{(iii)}$$

Thus, the flow per unit time av_1 is proportional to the square root of the difference $p_1 - p_2$. Two tubes, c and d, lead to a pressure gauge and clockwork arranged to integrate $(p_1 - p_2)^{\frac{1}{2}}$, and provided with dials giving direct readings of the amount of water which has passed.

CHAPTER VIII.

CHEMICAL PHYSICS.

196. Homogeneous substances. — A homogeneous substance is one which is similar in all its parts. Organic tissue, living or dead, is non-homogeneous. Any non-homogeneous substance must be reduced to one or more homogeneous substances before any detailed chemical or physical study can be made of it.

197. Elementary and compound substances. — A homogeneous substance can, in many cases, be broken up into two or more distinct homogeneous component parts, each having characteristic properties. Such a substance is called a *compound substance* or a *chemical compound.* The component parts may themselves be compound. Substances which have never yet been broken up in this way are called *elementary substances* or *chemical elements*, of which about seventy are at present known.

198. Conservation of matter. — The sum of the masses of the component parts of a compound substance is equal to the mass of the compound. That is, the total amount of matter, as measured by its mass, remains constant whatever changes it may be made to undergo.

199. Law of constant proportions. — The masses of the component parts of a given compound bear a constant ratio to each other, called the *combining ratio* of the two parts. A homogeneous substance of which the component parts are not in a fixed ratio is called a *mixture.* Thus aqueous solutions, air, etc., are mixtures. Two elements often have several distinct combining ratios. Consider several chemical elements a, b, c, $\cdots$; let b_1, b_2, b_3, $\cdots$, c_1, c_2, c_3, $\cdots$, and d_1, d_2, d_3, $\cdots$, be the respective masses of these elements which can combine with a given mass a_1 of the element a. Then $a_1 : b_1$, $a_1 : b_2$, $a_1 : b_3$, $\cdots$ are the combining ratios of a and b; $a_1 : c_1$, $a_1 : c_2$, $a_1 : c_3$, $\cdots$ are the combining ratios of a and c, and so on. Further, **the actual combining ratios of b and c are found to be included in the set $b_1 : c_1$, $b_1 : c_2$, $b_1 : c_3$, $b_2 : c_1$, $b_2 : c_2$,** etc., although all of these ratios may not occur.

200. Law of multiple proportions. — The masses b_1, b_2, b_3, $\cdots$ of an element b which can combine with a given amount a_1 of an element a are *multiples* of some one number, so that the ratios $b_1 : b_2$, $b_1 : b_3$, $\cdots$ are rational fractions.

201. Molecular theory.* — The above laws of chemical combination, as well as other experimental laws to be given hereafter, are clearly represented if we assume that each chemical element consists of ultimate similar particles of equal mass called *atoms*, and that the atoms of two or more elements in a compound are arranged in similar groups called *molecules*. Thus nitrogen and oxygen combine to form N_2O, NO, N_2O_3, NO_2, and N_2O_5.

The combining ratios of these various compounds are 2802 : 1588, 1401 : 1588, 2802 : 4764, 1401 : 3176, and 2802 : 7940 respectively. These numbers are multiples of 1401 on the one hand and of 1588 on the other. The number 1401 is called the *atomic weight* of nitrogen and the number 1588 is called the atomic weight of oxygen, the atomic weight of hydrogen being chosen arbitrarily as 100.

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* This molecular theory proves very useful throughout the subject of chemistry, in the theory of gases and solutions, in the theory of heat, in the theory of crystallization, and in the theory of electrolysis. The student should look to this *use*. All elaborations of the theory are suggested by and explanatory of obscure chemical and physical phenomena, and until the student takes up the study of such phenomena, it is worse than useless to dwell upon the details of the theory.

202. States of aggregation; solids; liquids; gases. — A *solid* is a substance which has more or less rigidity of form or which can withstand, at rest, a shearing stress. A *fluid* is a substance which has no rigidity of form. (See Art. 166.) A *liquid* is a fluid capable of having a free surface and of which the volume is definite. A *gas* is a fluid of which the volume is limited only by that of the closed containing vessel.

The phenomena of heat indicate, as we shall see, that the molecules of all substances are in a state of erratic motion more or less violent. In solids it is very likely that each molecule in its motion does not move to any sensible distance from its mean position, but, on the whole, retains its proper place in the molecular structure (crystalline) of the solid. In liquids and gases each molecule wanders throughout the mass of the fluid, as is shown by the phenomena of diffusion and electrolysis.

203. Crystallization. — *Crystalline substances.* Some liquids (also some gases) under proper conditions, *e.g.* when cooled to a certain temperature, begin to deposit solid parts called *crystals,* which are of regular form and bounded (usually) by plane faces. This phenomenon is called *crystallization.* Solids formed in this way are said to be *crystalline.* The physical properties of crystalline solids, such as *cleavage,* and their elastic and electrical properties, indicate definite molecular *structure.* Indeed the phenomenon of crystallization is to be thought of as a definite grouping of the molecules as they pass from the freedom of the liquid state to the solid state. (See Art. 210.)

204. Amorphous substances. — Other liquids under proper conditions, *e.g.* when slowly cooled, become more and more viscous, until to all appearances solid. Such substances have no definite melting point (see Art. 242), they give no evidence of molecular structure, and are called *amorphous substances.* Most amorphous solids are very slightly viscous, and exhibit other properties of liquids. For example, glass is known to be an electrolyte, a fact which indicates that some of its molecules or atoms must wander throughout its mass.

205. Crystalloids; colloids. — Substances which crystallize upon solidification are called *crystalloids.* Some instances, sugar for example, may be either amorphous or crystalline according to the conditions under which the solidification takes place. Some substances, such as glue and albumen, seem to be totally incapable of crystallization. Such substances are called *colloids.*

206. Law of constant angles * (Romé de l'Isle). — During the growth of a crystal the angles between the various faces remain unchanged, and for a given substance these angles are invariable. The amount of deposition on a given face, *i.e.* the growth of that face, varies greatly with circumstances. Therefore the *directions only* of the various faces are significant.

207. Axes of reference; intercepts of a face. — It is convenient to choose as axes of reference three lines parallel respectively to *any three* † *edges of a crystal which do not lie in one plane.*

Let the plane of a crystal face be extended until it cuts these X, Y, Z axes of reference at distances a, β, and γ respectively from the origin. These distances, a, β, and γ, are called the *intercepts* of the face.

208. Axial parameters of a crystal. — *Indices of a face.* It is convenient in crystallography to express the intercepts a, β, and γ of a face in terms of three chosen ‡ lengths a, b, c in the

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* The following discussion of crystallography cannot be followed by the beginner without the help of *crystal models.* Actual crystals are not so good, since the eye scarcely perceives *symmetry of angles* except it be accompanied by *symmetry of dimensions,* which is easily provided in models, but only occurs accidentally in actual crystals. The reader is referred to E. S. and J. D. Dana, A Text-book of Mineralogy, Wiley and Sons, 1886, for a full discussion of crystallography.

† It is evident that these axes may be other than rectangular. The choice of axes of reference here indicated greatly simplifies the statement of the law of rational indices. (See Art. 209.)

‡ For manner of choice see Art. 209.

directions of the respective axes of reference, such that $\alpha = la$, $\beta = mb$, and $\gamma = nc$.

The chosen lengths a, b, and c are called the *axial parameters* of the crystal, and l, m, and n are called the *indices* of the given face. Since the direction only of a face is significant, the intercepts la, mb, and nc may be divided by any number, say a; this is equivalent to displacing the face parallel to itself; the intercepts then become l, $m \cdot \frac{b}{a}$, and $n \cdot \frac{c}{a}$. The permissible values of the indices l, m, and n for the various faces of a crystal are determined from the *law of symmetry* (Art. 210) and the *law of rational indices* (Art. 209), as we shall see, so that a *given crystal is completely characterized by the values of* $\frac{b}{a}$ *and* $\frac{c}{a}$, *the angles between the axes of reference, and by its degree of symmetry.* The values of $\frac{b}{a}$ and $\frac{c}{a}$, and the angles between the axes, are called the *elements of a crystal*, and the degree of symmetry (there are six degrees) determines to which one of six *systems* the crystal belongs.

209. Law of rational indices (Malus). — It is always possible to choose $\frac{b}{a}$ and $\frac{c}{a}$ (Art. 208) for a given crystal so that for every face of the crystal the indices l, m, and n are integers.

The intercepts of a number of faces of a given crystal upon the axes of reference being determined by observation, the numerical values of $\frac{b}{a}$ and $\frac{c}{a}$ are determined upon the condition* that l, m, and n shall be integers for every face. There is a series of values of $\frac{b}{a}$ and $\frac{c}{a}$ which satisfy this condition, the smallest values being chosen.

210. The molecular theory of crystallization. — All the experimental facts concerning crystals are clearly represented if we imagine a crystal to be built up of molecules which are deposited

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* The possibility of this condition in every case is Malus' discovery.

in *layers* upon the various faces. The molecules in each layer being arranged in *rows*, each row being the line of intersection of two possible faces of the crystal.

The **axes of reference** are thus in the directions of *rows* of molecules. If the spacing of the molecules in these axial rows is imagined to be proportional to the axial parameters a, b, c of the crystal, then the indices l, m, n of any face (or layer of molecules) must be integers. The most familiar example of this, in two dimensions, is afforded by a field of Indian corn in which the hills are in rows both ways. The hills then form diagonal rows. Choose any two rows as axes of reference. Then any other row not passing through the origin has *intercepts* on these axes and these intercepts are integral multiples of the distances between adjacent hills in the respective axial rows.

211. Symmetry of crystals. — Whenever any two edges of a crystal are at right angles to each other, or whenever any three edges are mutually rectangular, it is conducive to simplicity to choose such edges as axes of reference.

Crystals are classified according to their degree of symmetry, into six systems, as follows:

Orthogonal systems, to which all crystals belong, which have, possibly,* three mutually perpendicular edges, which are chosen as axes of reference. There are three such systems:

(*a*) *The cubic* or *isometric system.* In this system the faces are similarly grouped about each of the three orthogonal axes. Therefore $a=b=c$ (Art. 208), and these crystals are symmetrical to *nine* planes, of which three are the axial planes xy, yz, and zx, and the other six are diagonal thereto.

(*b*) *The tetragonal* or *dimetric system.* In this system the faces are similarly grouped with respect to each of two only of the orthogonal axes. Therefore $a \lessgtr b=c$, and these crystals are

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* An edge or a face may be possible on a crystal of a given substance without actually occurring. That is, certain layers of molecules in a crystal may not always show as faces.

symmetrical to *five* planes, of which three are the axial planes and the other two are diagonal planes intersecting along the axis *a*.

(*c*) *The orthorhombic* or *trimetric system.* In this system the faces are dissimilarly grouped about the three orthogonal axes. Therefore *a*, *b*, and *c* are all different, and these crystals are symmetrical to *three* planes, which are the axial planes.

Inclined systems, to which all crystals, except hexagonal crystals, belong which do not have, possibly, three mutually perpendicular edges. There are two such systems:

(*d*) *Monoclinic system.* Crystals belonging to this system have, possibly, one edge which is at right angles to two others. In this system *a*, *b*, and *c* are unequal.* Monoclinic crystals are symmetrical to *one* plane which is the plane of the oblique axes.

(*e*) *Triclinic system.* Crystals belonging to this system have no two possible edges at right angles. Therefore the axes of these crystals are all oblique, and *a*, *b*, and *c* are unequal.† Triclinic crystals have no plane of symmetry; the only symmetry they possess consists in the fact that diametrically opposite parts are similar.

(*f*) *The hexagonal system.* Crystals which have possibly three edges 60° apart in one plane, and another edge at right angles thereto, and of which the faces are similarly grouped with respect to the three coplanar edges, are classified together as *hexagonal* crystals. Such crystals have four axes. The three coplanar axes are called the *lateral* axes. The *lateral* axial parameters are equal.‡

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* A crystal admitting of monoclinic axes, but of which the arrangement of the faces requires two or more of *a*, *b*, and *c* to be equal, falls into one of the previously mentioned systems.

† A crystal admitting of triclinic axes, but of which the arrangement of the faces requires two or more of *a*, *b*, and *c* to be equal, falls into one of the previously mentioned systems.

‡ On account of the similar grouping of the faces with respect thereto.

There are two subdivisions of *hexagonal* crystals:

(1) Those which are symmetrical to seven planes; namely, the plane of the lateral axes which is called the *basal plane*, and six planes perpendicular to the basal plane, three of which include the lateral axes and three, called the diagonal planes, are midway between them.

(2) Those which are symmetrical to four planes; namely, the basal plane and the diagonal planes just mentioned.

212. Solution. — Some solids when placed in a liquid, for example, sugar in water, are disaggregated; the molecules of the solid become free and wander about through the liquid. The solid is said to *dissolve* in the liquid and the mixture, which soon becomes homogeneous, is called a *solution.* Gases are, in many cases, absorbed by a liquid in indefinite proportions (Art. 199), and one liquid often mixes with another. Such mixtures are also called solutions.

213. Solubility. — When a liquid at a given temperature has dissolved the greatest possible amount of a substance, the resulting solution is said to be *saturated.* The mass of a substance per unit mass of liquid in a saturated solution at a given temperature is called the *solubility* of the substance at that temperature. The solubility of solids in nearly every case increases rapidly with rise of temperature; the solubility of a gas ordinarily decreases with rise of temperature.

214. Supersaturation. — If a saturated solution of a substance be cooled slowly, the substance is ordinarily slowly deposited, usually as crystals. The beginning of this action requires a nucleus such as a particle of dust, a sharp point of a submerged body, or a crystal of the substance. Such nucleus being absent, the solution becomes *supersaturated* as it is cooled. A supersaturated solution is in a state of unstable equilibrium and in many cases a very slight disturbance of any kind upsets it and causes the deposition of crystals. Liquids which are on the point of freezing behave as saturated solutions.

215. Diffusion. — When two liquids which are miscible, for example, pure water and an aqueous solution of any salt, are brought carefully together so as not to mix by convection, it is found that in the course of time they become uniformly mixed. This phenomenon is called *diffusion;* it shows that the molecules of liquids wander as mentioned in Art. 202. Gases mix in the same way, but with greater rapidity.

216. Coefficient of diffusion. — *The rate of diffusion of a substance in solution is proportional to the concentration gradient of the solution* (Graham's Law). *Explanation:* Consider a vessel containing water, with crystals of soluble salt at the bottom. This salt will dissolve, forming a solution which grows less concentrated towards the top of the vessel. The concentration is reckoned as grams of salt per gram of solution. Consider the diffusion across an imaginary horizontal plane area a. Let Δc be the difference in concentration at two points near the plane, distant Δx the one above the other, then $\frac{\Delta c}{\Delta x}$ or $\frac{dc}{dx}$ is called the *concentration gradient aba.* By rate of diffusion is meant the number of grams of salt which diffuse across unit area per unit time. Graham found this rate to be approximately proportional to $\frac{dc}{dx}$. The proportionality constant is called the *coefficient of diffusion* of the salt in the given solvent.

217. Diffusion through membranes; osmosis. — When two liquids are separated by a porous membrane, it is found that one of them — perhaps the one which wets the membrane with the greater facility — diffuses through the membrane faster than the other. This phenomenon is called *osmosis.* If the liquids are two different aqueous solutions, the salts, being different, pass through the membrane with unequal facility.

Colloids pass through membranes very slowly and crystalloids with much greater rapidity.

218. Semi-permeable membranes; osmotic pressure.—Some membranes allow water to diffuse through them quite freely, while salts in solution pass with difficulty. Such are called *semi-permeable* membranes. If a solution of a salt be separated from pure water by a semi-permeable membrane, water will pass through the solution until the pressure of the solution exceeds that of the water by a certain amount. This excess of pressure is called the *osmotic pressure* of the solution. The conception of osmotic pressure plays an important part in the theory of solutions.

PART II. — HEAT.

INTRODUCTORY REMARKS CONCERNING HEAT.

219. The principle of the conservation of energy is fully established (as was clearly apprehended by Newton himself) for a system conforming to Newton's laws of motion and to which the conception of *potential energy* applies; that is, a system such that the work required to carry it from one position configuration to another is independent of the intermediate stages through which the system is made to pass. In treating moving systems of finite dimensions it is convenient to ignore a very important class of motions and mutual force actions; namely, the relative motion of, and mutual force actions between, the *small parts* of the moving bodies, or in other words *molecular* motion and *molecular* forces. In such case it often occurs that the apparent energy of a closed system *decreases*, the apparent decrease being due to an *increase* of the energy of the ignored motion. Such apparent decrease of energy always occurs when inelastic bodies collide, and in moving systems affected by friction. It is perhaps conceivable that a system of bodies might be completely treated by the ordinary methods of mechanics, using Newton's laws of motion and explicitly considering each individual particle, or atom, in the system. The enormous number of molecules in bodies of ordinary size leads, according to the theory of probability, to a high degree of constancy in their average behavior, and this constancy of average behavior forms the basis of a simple and efficient mathematical theory called **Thermodynamics** which enables the phenomena depending upon molecular energy to be treated in a complete manner without explicit reference of any kind to molecules.

Both laws of thermodynamics are essentially the result of the constancy of the average behavior of molecules; the peculiar methods of thermodynamics do not appear, however, until after the consideration of the second law. Let the following serve as a partial example. Consider a given amount of air in a vessel. The average distance between molecules, or what amounts to the same thing, the *volume* occupied by the gas, is a complete specification of the *position configuration* of the system. The velocities of individual molecules are in all directions and of all magnitudes, but since the energy of the gas is definite, the average square of the molecular velocities must have a definite value, and the *temperature* of the gas, which is known to be proportional to this average square, is a complete specification of the *velocity configuration* of the system. Thus **temperature** and **volume** together completely specify the state of molecular motion of a given amount of gas. The same is true of any homogeneous liquid, and to a large extent also of solids.

CHAPTER IX.

220. Definition of heat. — *The energy stored in a body by virtue of the state of its molecular motion* is called heat.* For example, energy communicated to a gas goes to increase the molecular kinetic energy; energy communicated to ice at the melting-point causes some of the ice to melt, and the energy is used, perhaps entirely, to alter the molecular configuration of the ice which melts; energy communicated to water warms it and is used in part to alter the molecular configuration of the water and in part to increase the molecular kinetic energy.

221. First law of thermodynamics. — *The apparent loss of energy in a closed system is always accompanied by the generation of an amount of heat* (see below for methods for measuring heat) *which is exactly equivalent thereto.*

This law is to some extent involved in the above definition of heat; however, as a statement of observed fact it serves to establish the definition of heat upon a sound basis. The ultimate significance of the law is that molecules obey Newton's laws of motion, that the conception of potential energy is universally applicable, and that the principle of the conservation of energy is entirely general. Compare Art. 219.

222. Effects of heat. — The effects of heat upon bodies are:

(*a*) To increase their temperature. This increase of temperature is generally accompanied by expansion.

(*b*) To change their state of molecular aggregation — if enough heat be applied disaggregation always results.

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* The term *molecular motion* here and in subsequent articles refers to the "relative motions of small parts" of a body of which mention has been made in the preceding introductory remarks concerning heat (Art. 219).

(*c*) To dissociate the atoms in the molecules of chemical compounds.

(*d*) To produce certain electrical phenomena (see chapters on electricity).

223. Definition of equal temperatures.* — Two bodies are said to be at the same temperature when no exchange of heat takes place between them when they are placed in contact.

NOTE. — Two bodies *A* and *B* of which the respective temperatures are equal to the temperature of a third body are found to suffer no exchange of heat when placed in contact with each other. This is by no means axiomatic.

NOTE. — The temperature of a body *A* is *greater* than the temperature of a body *B*, when upon bringing *A* and *B* into contact, heat passes from *A* to *B*.

224. Gay Lussac's law. — Consider a glass bulb (Fig. 117) containing a gas. This gas is separated from the outer atmosphere by means of a column of mercury.

The temperature of the vessel being changed and the pressure upon the contained gas being maintained constant, the volume of the gas will change. By rise of temperature the mercury column will recede before the expanding gas, and *vice versa.* It is found when different gases are experimented upon that by common change of temperature the change of volume will be the same for all. This is the fact expressed in the first clause of what is known as Gay Lussac's or Charles' law, viz.:

Fig. 117.

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* A rigorous numerical definition of temperature will be given under the Second Law of Thermodynamics.

(*a*) *When various gases are heated under constant pressure, they all suffer the same expansion.*[*]

It has been stated in a previous article (167) that in a true gas the volume is inversely proportional to the pressure to which it is subjected (Boyle's law). We may therefore imagine the experiment which we are considering to be carried out under conditions such that the volume of the gas is maintained constant, by continual variations of pressure, and the relations between the pressure and temperature are noted.

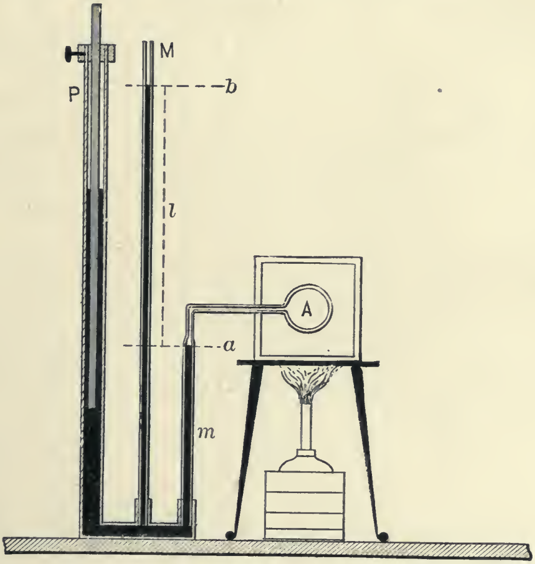

Fig. 118.

The apparatus would then take the form shown in Fig. 118. In this diagram *A* is the bulb of gas to be subjected to change of temperature. Its volume is controlled and maintained constant by means of a manometer of which *M* is the open arm

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[*] In the second clause of his law Gay Lussac gave the numerical value of the coefficient of expansion of a gas. His value, however, was incorrect and it is now known that $p : p' = 273 : 373$ or $v : v' = 273 : 373$; p and p' being the pressures of a constant volume of gas at ice temperature and steam temperature (Art. 226) respectively; and v and v' are the volumes at ice temperature and steam temperature respectively, of a gas of which the pressure is maintained constant.

while P is an adjusting tube containing a cylindrical rod of iron which may be submerged to any required extent in the mercury, thus forcing the liquid over into M and m, and varying the pressure. By means of this apparatus various gases may be tested and the pressures (as measured by means of the length l of the vertical column between a and b, plus 1 atmosphere) necessary to maintain these gases at constant volume when change of temperature occurs, may be observed. The results will give us another statement of Gay Lussac's law which is more useful for the purpose of our present discussion, viz.:

(*b*) *All gases when heated without change of volume, follow the same law of increase of pressure with temperature.*

225. Provisional definition of the ratio of two temperatures. — The second form of Gay Lussac's law affords a convenient* basis for the following provisional definition of the ratio of two temperatures.

The ratio of two temperatures (provisionally defined) is the ratio of the pressures of a constant volume of gas at those respective temperatures. That is, p and p' being the pressures of a given volume of gas at temperatures t and t' respectively, we have by definition

$$\frac{t}{t'} = \frac{p}{p'}. \tag{132}$$

This provisional definition will be found to coincide with the thermodynamic definition of temperature.

A definite numerical value cannot be assigned to any temperature until an arbitrary value has been assigned to some standard temperature or until an arbitrary numerical value has been assigned to the *difference* between two standard temperatures.

226. Standard temperatures. — Experiment shows that the temperature of pure melting ice at a given pressure and that the minimum temperature of pure steam at a given pressure

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* Convenient because not dependent upon any *particular* gas.

are invariable. These at a pressure of 76 cm. are chosen as the standard temperatures of reference in thermometry and are called the "ice point," I, and "steam point," S, respectively.

Of the various effects of heat, mentioned in Art. 222, the expansion which accompanies rise of temperature has been found most suitable and convenient for the indication of temperatures. Of the many substances available in thermometry, the most useful are air, at constant volume, and mercury. The instruments based upon the thermometric behavior of these two materials are termed, respectively, the *air thermometer* and *the mercury thermometer*. They are discussed in Art. 203 *et seq.*

227. The air thermometer.—A bulb A (Fig. 119) of hard glass, filled with dry air, communicates, by means of a tube of fine bore, with the short arm of a syphon barometer BB. Since there is a vacuum at V, the distance l affords a measure of the pressure of the air in A. An open mercury reservoir R, which can be moved up or down, connects with the barometer by means of a flexible tube, and serves to bring the surface of the mercury at a to a marked point near the opening of the fine tube. The bulb is placed in a bath of melting ice, and the pressure p in A is observed. The bulb is then placed in a steam bath at standard pressure, and the pressure p' in A is again observed. Then from equation (132) we have:

$$\frac{I}{S}=\frac{p}{p'}. \qquad (132\ bis)$$

It is customary to choose

$$S-I=100. \qquad (133)$$

From these two equations, S and I

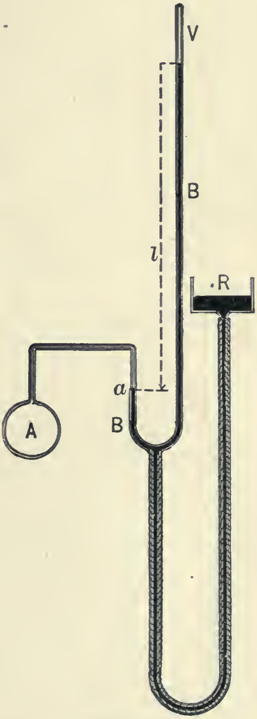

Fig. 119.

may be calculated. It is found in this way, that $S = 373$, and $I = 273$, see Art. 224, footnote.

The values of I and S having been determined as above, any other temperature T is determined as follows: The bulb A is placed in a steam bath at standard pressure, and the pressure p' in A is observed. The bulb is then placed in the region of which the temperature T is to be determined, and the pressure p'' in A is observed. Then from (132) we have

$$\frac{T}{S} = \frac{p''}{p'},$$

or

$$T = \frac{p''}{p'} S. \tag{134}$$

NOTE. — Temperatures determined in this way by the air thermometer are called **absolute temperatures**, inasmuch as the zero temperature on this scale is a temperature at which the pressure of a gas becomes zero, and is probably the absolute zero of temperature. The temperature reckoned from the ice point, *i.e.* $T - I$, is nearly the same as temperature indicated by an ordinary centigrade mercury thermometer.

228. The mercury thermometer. — A glass tube AB (Fig. 120) of fine uniform bore, having a bulb on one end, is completely filled with mercury at a temperature somewhat above the steam point and is hermetically sealed at A. As it cools, the mercury contracts more rapidly than the glass, and thus only partially fills the stem. The whole is placed in an ice bath, and the position of the surface of the mercury in the stem is marked at I. In a similar manner the steam point is marked at S.

In **the centigrade scale (Celsius)**, which is the scale universally used in scientific work, the distance SI is subdivided into one hundred equal parts, which divisions are continued above S and below I. These marks are numbered upwards, beginning at I, which

Fig. 120.

is number zero. The marks below I are numbered negatively from I.*

Any temperature is specified by giving the number of the mark at which the mercury stands when the thermometer is brought to that temperature. For example, 65° C., read *sixty-five degrees centigrade*, is the temperature at which the mercury in a mercury thermometer stands at mark number 65 of the centigrade scale; 97° F., read *ninety-seven degrees Fahrenheit*, is the temperature at which the mercury stands at mark number 97 of the Fahrenheit scale.

Temperatures, as indicated by a mercury thermometer, are called *mercury-in-glass* temperatures.

229. Formal definition of mercury-in-glass temperature. — Consider two mercury-in-glass temperatures t and t'; let v and v' be the volumes of the bore of a thermometer stem from I to t and from I to t' respectively; then

$$\frac{t}{t'}=\frac{v}{v'}. \tag{135}$$

This is evident when we consider that t and t' are the *numbers* of equal divisions of a tube of uniform bore between I and t and I and t' respectively.

The relation between air-thermometer temperatures and mercury-in-glass temperatures, where the glass is of the variety known as Jena normal glass, is shown in Fig. 121. The curve is drawn from measurements by Wiebe.

The abscissas are temperatures reckoned from the ice point. Ordinates, measured from the horizontal line marked "*air*," indicate the corrections which must be applied to mercury-in-glass temperatures to give the corresponding air-thermometer temperatures.

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* The only other scale of which mention need be made is that of Fahrenheit, in which the distance SI is subdivided into one hundred and eighty equal parts, which divisions are continued above S and below I. These marks are numbered upwards, beginning with the thirty-second mark below I, which is number zero. The marks below zero are numbered negatively.

Remark. — All liquids and solids expand irregularly; therefore the mercury thermometer must be considered strictly as a scheme for *naming* temperatures. For purposes of simple specification, mercury-in-glass temperatures are entirely satisfactory.

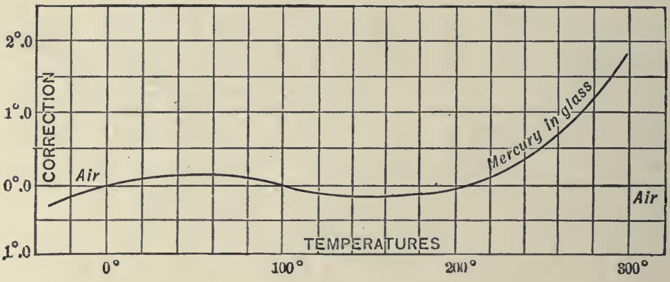

Fig. 121.

From Fig. 121 it is evident that no great errors will result from the use of the indications of a mercury-in-glass thermometer as true measures of temperature.

230. Standardization of a mercury thermometer. — The indications of a mercury thermometer deviate from true mercury-in-glass temperatures, (1) because of irregularities in the bore of the stem, (2) because of errors in the location of the steam and ice points, (3) because of irregularities in the divisions, and (4) because of the use of glass having other than standard composition.

In the following discussion the thermometer is assumed to be of standard glass and the thermometer scale is considered to be simply a *scale of equal parts*.

(*a*) **Observational data.**

(1) The scale reading, a (Fig. 122), when the thermometer is in an ice bath.

(2) The scale reading f, when the thermometer is in a steam bath at observed pressure $p^{\text{mm.}}$. The true temperature T of the steam bath is then

$$T = 100^\circ \text{ C.} - .0375\,(760 - p), \qquad (136)$$

providing p is not very different from $760^{\text{mm.}}$. This equation (136) results from the observed fact that the temperature of a steam bath in the neighborhood of $760^{\text{mm.}}$ pressure, decreases .0375 degree centigrade for each mm. decrease of pressure.

(3) Let v be the volume of the bore between a and f. Let a portion of the mercury filament be detached, of volume roughly equal to, say, $\frac{1}{4}v$, so that placed end to end four times it may reach very nearly from a to f. Let this filament be placed with its lower end at a and let the position of its upper end b be noted. Then let it be placed with its lower end at b, and let its upper end c be read, and so on. The readings thus obtained are b, c, d, and e.

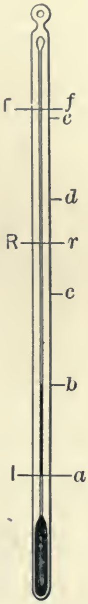

Fig. 122.

(b) **Calculation** of true mercury-in-glass temperature R corresponding to a given reading r. Assume the bore of the tube to be uniform throughout each section ab, bc, cd, and df. Let x be the volume of the detached filament; then

$$v = 4x + \frac{f-e}{e-d}x; \tag{137}$$

for since the tube is assumed uniform between d and f, the volume of the portion ef of the bore is $\frac{f-e}{e-d}x$, the scale being one of equal parts. Similarly, the volume v' of the bore between a and r is

$$v' = 2x + \frac{r-c}{d-c}x. \tag{138}$$

From equation (135) we have $\frac{T}{R} = \frac{v}{v'}$, whence, substituting values of v and v' from (137) and (138) and solving for R, we have

$$R = \frac{2 + \frac{r-c}{d-c}}{4 + \frac{f-e}{e-d}}T, \tag{139}$$

from which x has disappeared. This formula differs according to the position of the reading r. The assumed uniformity of the bore of the stem throughout the sections *ab*, *bc*, etc., leaves slight outstanding errors; these are, however, very much smaller than would result from an assumed uniformity of bore throughout. Such outstanding errors may be in turn eliminated by calibrating the respective sections.

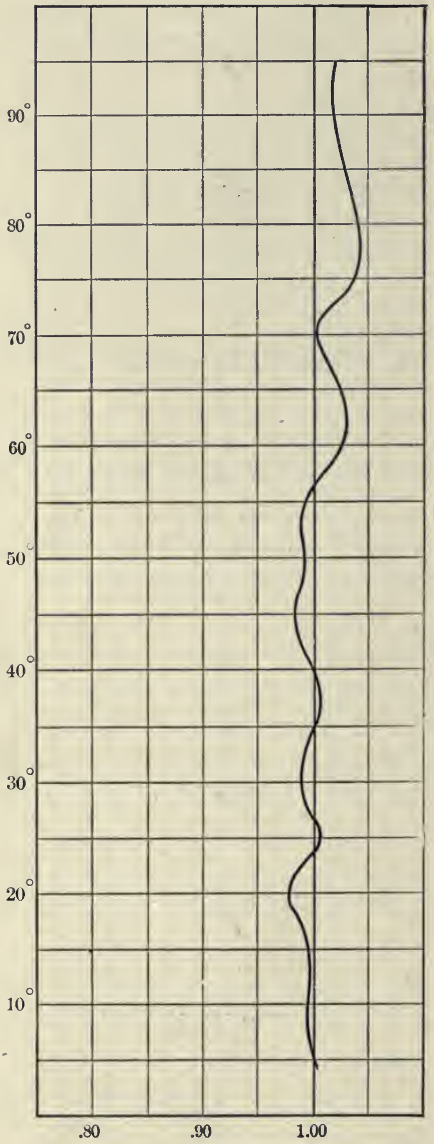

Fig. 123.

How important such calibrations of the tubes of mercury thermometers, which are to be used in operations, really are, will be seen from Fig. 123, in which are shown graphically the irregularities of the bore of an excellent thermometer between 5° C. and 95° C. Ordinates are distances from the ice point, and abscissas are cross-sections of the bore, in terms of that of the average cross-section taken as unity. It will be seen that the ratio of the largest cross-section measured to the smallest was $\frac{1.042}{.976} =$ 1.068 (nearly 7 per cent).

231. Expansion. — In a preceding article (222), various effects of heat were mentioned. We are now ready to consider one of these; *i.e.* the changes of dimension which accompany rise of temperature.

(*a*) **Linear expansion.** — The ratio $\frac{l_t}{l_0}$ of the length l_t of a body at temperature t to its length l_0 at 0° C., is a function of t, and its value is unity when $t=0°$ C. Therefore, by Taylor's theorem, $\frac{l_t}{l_0}=1+\alpha t+\alpha' t^2+\alpha'' t^3+$etc., or,

$$l_t=l_0(1+\alpha t+\alpha' t^2+\alpha'' t^3+\text{etc.}), \tag{140}$$

in which α, α', α'', etc., are constants.

When t is not large, equation (140) may be written

$$l_t=l_0(1+\alpha t). \tag{141}$$

The quantity α is called the coefficient of linear expansion of the substance. The coefficient of linear expansion of a bar is determined by measuring its length, l_0, at zero, and its length, l_t, at an observed temperature t, in terms of that of a bar kept at constant temperature; α is then calculated from equation (141).

(*b*) **Cubic expansion.** — Let v_0 be the volume of a body at 0° C., and v_t its volume at temperature t; then, as in equation (140), we have

$$v_t=v_0(1+\beta t+\beta' t^2+\beta'' t^3+\text{etc.}), \tag{142}$$

in which β, β', β'', etc., are constants.

When t is not large, equation (142) may be written

$$v_t=v_0(1+\beta t)\cdots. \tag{143}$$

The quantity β is called the coefficient of cubic expansion of the substance. The value of β may be calculated from (143), the volumes v_0 at zero, and v_t at observed temperature t having been observed.

(*c*) **Proposition.** — The coefficient of cubic expansion of a substance is equal to three times its coefficient of linear expansion; that is

$$\beta=3\alpha. \tag{144}$$

Proof. — Consider a cubical portion of a substance, of which the edge is of length l_0 at 0° C.; at t° C. the length of its edge is $l_0(1+\alpha t)$, so that $v_t = l_0^3(1+\alpha)^3 = l_0^3(1+3\alpha)$, since terms in $\alpha^2 t^2$ and $\alpha^3 t^3$ are negligible. Writing v_0 for l_0^3, in this expression we have $v_t = v_0(1+3\alpha)$. Comparing this with (143) we have $\beta = 3\alpha$.

(*d*) **Variation of density with temperature.** — Dividing the mass of a body by v_0 and v_t, we have its densities d_0 and d_t at zero and at t° C. respectively, so that from (143) we have

$$d_t = d_0 \frac{1}{1+\beta t},$$

or, since βt is small,

$$d_t = d_0(1-\beta t). \qquad (145)$$

232. Regnault's method for determining the coefficient of cubic expansion of a liquid. — Two tubes, A and B, open at top, and connected by an air tube C at lower end, are filled with the liquid as shown in Fig. 124. The tube A is placed in a bath at temperature t, and B in a bath at temperature zero. The vertical distances l_0 and l_t are measured. Since the pressure in C is equal to $l_0 d_0 g$ and to $l_t d_t g$, the ratio $\frac{l_0}{l_t}$ is equal to $\frac{d_t}{d_0}$. Substituting $\frac{l_0}{l_t}$ for $\frac{d_t}{d_0}$ in equation (145), β may be calculated. The reader will see that by this method, also, the series of coefficients β, β', β'', etc., in equation (142) may be determined. If it is desired to determine, say, two of these coefficients, then the ratio $\frac{v_t}{v_0}$ must be determined for at least two temperatures t.

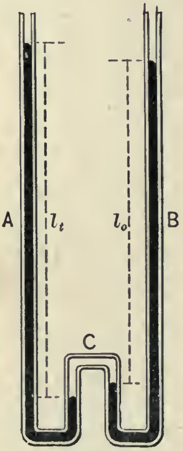

Fig. 124.

The following table gives the density of mercury determined by Regnault's method, and also the volume of one gram of that liquid at various temperatures.

TABLE III.

TEMPERATURE.	WEIGHT OF 1 C.M³.	VOLUME OF 1 GRAM.	TEMPERATURE.	WEIGHT OF 1 C.M³.	VOLUME OF 1 GRAM.
0° C.	13.5956	0.073553	190°	13.1385	0.076112
10°	.5709	.073687	200°	.1150	.076249
20°	.5463	.073821	210°	.0915	.076386
30°	.5218	.073954	220°	.0680	.076523
40°	.4974	.074088	230°	.0445	.076660
50°	.4731	.074222	240°	.0210	.076799
60°	.4488	.074356	250°	12.9976	.076937
70°	.4246	.074490	260°	.9742	.077076
80°	.4005	.074624	270°	.9508	.077215
90°	.3764	.074759	280°	.9274	.077355
100°	.3524	.074893	290°	.9041	.077495
110°	.3284	.075028	300°	.8807	.077635
120°	.3045	.075162	310°	.8573	.077776
130°	.2807	.075297	320°	.8340	.077918
140°	.2569	.075432	330°	.8107	.078060
150°	.2331	.075568	340°	.7873	.078202
160°	.2094	.075703	350°	.7640	.078345
170°	.1858	.075839	360°	.7406	.078489
180°	.1621	.075975			

233. The indirect determination of the expansion of water and of other liquids. — The expansion of a liquid may be studied by Regnault's method (Art. 232), or indirectly by measuring its volume in a vessel of glass of which the expansion has been previously determined.

The volumes v_0 and v_t of a glass vessel at zero and t° C., respectively, can be easily determined from the observed net weight of mercury contained by the vessel at the respective temperatures and the known densities of mercury at zero and t° C. Equation (143) then enables the calculation of β.

The behavior of water, which shows marked peculiarities, *contracting* when heated from 0° to 4°, at which temperature its density is a maximum, and expanding as the temperature rises above the latter value, is indicated in Fig. 125, in which ordinates are volumes and abscissas are temperatures. The data for this curve are taken from the accompanying table, which contains densities and relative volumes of water at temperatures between – 10° C and + 100.° C.

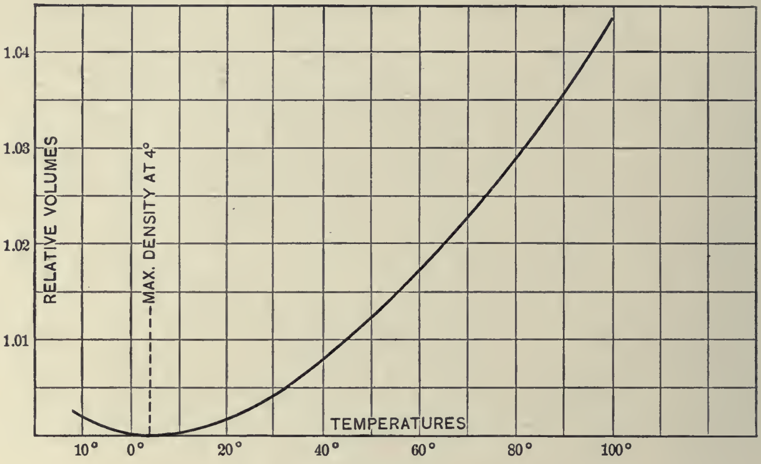

Fig. 125.

TABLE IV.

Relative Densities and Volumes of Water (referred to 4° C.)

Temperature.	Density.	Volume.	Temperature.	Density.	Volume.
– 10° C.	0.99815	1.00186	20°	0.998252	1.001751
– 8°	0.99869	1.00131	25°	0.997098	1.002911
– 6°	0.99912	1.00088	30°	0.995705	1.004314
– 4°	0.99945	1.00055	35°	0.994098	1.005936
– 2°	0.99970	1.00031	40°	0.99233	1.00773
0°	0.999874	1.000127	45°	0.99035	1.00974
+ 1°	0.999930	1.000071	50°	0.98813	1.01201
2°	0.999970	1.000030	55°	0.98579	1.01442
3°	0.999993	1.000007	60°	0.98331	1.01697
4°	1.000000	1.000000	65°	0.98067	1.01971
5°	0.999992	1.000008	70°	0.97790	1.02260
6°	0.999970	1.000030	75°	0.97495	1.02569
7°	0.999932	1.000068	80°	0.97191	1.02890
8°	0.999881	1.000119	85°	0.96876	1.03224
9°	0.999815	1.000185	90°	0.96550	1.03574
10°	0.999736	1.000265	95°	0.96212	1.03938
15°	0.999143	1.000858	100°	0.95863	1.04315

234. Determination of the cubic expansion of solids. — Comparisons of the volume of the solids to be tested are made at different temperatures by one of the following indirect methods.*

(*a*) The density of the solid is obtained at two temperatures by method of weighing in air and in water. From these data the ratio $\frac{v_t}{v_0}$ is derived.

(*b*) The solid is placed in a specific gravity flask the capacity of which is known at the two temperatures for which v_0 and v_t are to be found. The remainder of the bottle is filled at each temperature with a liquid, the coefficient of expansion of which is known, and the bottle is weighed. The weight of the empty bottle and of the solid having also been determined, the weight of liquid for which space remains at each temperature is thus found, thence its volume and the volume occupied by the solid.

(*c*) The substance in the shape of a bar is placed within a glass tube one end of which is closed. The open end of the tube is then drawn out into a narrow neck (Fig. 126) and the space within not occupied by the bar is filled with mercury. The mercury expelled when the tube is heated affords an indirect but very exact indication of the change of volume of the bar, and thus of the coefficient of expansion.

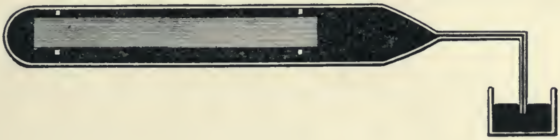

Fig. 126.

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* All of these methods depend upon a knowledge of the absolute expansion of liquids (see Regnault's method, Art. 232).

CHAPTER X.

CALORIMETRY.

235. Heat is measured by means of its effect in raising temperature. The substance used as a standard of comparison is water, and the unit in terms of which the result is given is called a thermal unit.

Thermal units. — The **gram-calorie**, which is the thermal unit chiefly used in scientific work, is defined as the amount of heat required to raise the temperature of one gram of water one centigrade degree. This amount of heat is slightly variable (see Art. 236), so that strict accuracy requires the initial and final temperatures to be specified. Thus the gramcalorie is understood to mean the amount of heat required to raise the temperature of one gram of water from 0° C. to 1° C., unless otherwise specified.

The **British thermal unit** is the amount of heat required to raise the temperature of one pound of water one Fahrenheit degree.

236. The mechanical equivalent of thermal units. — According to Joule, the British thermal unit is equivalent to 772 ft.-lbs. of energy. According to Rowland, 4.212×10^7 ergs of energy are required to raise the temperature of one gram of water from 5° C. to 6° C., and 4.179×10^7 ergs to raise the temperature of one gram of water from 19°.5 C. to 20°.5 C.

Remark. — Rowland's determination was made by driving a rotatory paddle about a vertical axis at a determined speed, in a vessel of water itself mounted on a vertical axis, and prevented from turning by cords passing over fixed pulleys to weights.

The torque exerted by the paddle is thus measured by the product of the *pull* of the cord into the lever arm thereof. This, multiplied by the angular velocity of the paddle and by the time, gives the energy expended; see equation (58). An accurate thermometer, projecting through the hollow axle of the paddle indicates the rise in temperature of the water.

237. The water calorimeter. — The number of thermal units H, required to raise the temperature of W grams of water from t' to t'', is

$$H = W(t'' - t'). \tag{146}$$

The **water calorimeter** is a vessel containing water arranged to absorb an amount of heat to be measured; the rise in temperature produced is observed by means of a thermometer, and the amount of heat is calculated from equation (146).

Sources of error in the use of the water calorimeter.

(*a*) A portion of the heat imparted to the calorimeter is used to raise the temperature of the containing vessel and the stirrer. The vessel and stirrer, being of the same material, are equivalent to an amount km of water, where k is the specific heat of the material, and m is their combined mass. $W + km$ is then to be used for W in equation (146). For accurate work, the water equivalent of the thermometer bulb must also be considered.

(*b*) When the calorimeter is cooler than room temperature, it absorbs heat from its surroundings, and *vice versa*. This source of error is, to a great extent, obviated by arranging that the initial temperature of the calorimeter be as much below room temperature as the final temperature is above room temperature, and that the duration of the experiment be as short as possible.

The calorimeter is best made of thin, polished metal, and surrounded by a thin, polished metal jacket, with air space between to reduce to a minimum the interchange of heat between the calorimeter and its surroundings. It usually takes the form of a cylindrical vessel *CC* (Fig. 127), mounted upon points of some material which is a poor conductor of heat, such as glass. *JJ* is the jacket; *M* is a solid, the thermal capacity of which is to be determined.

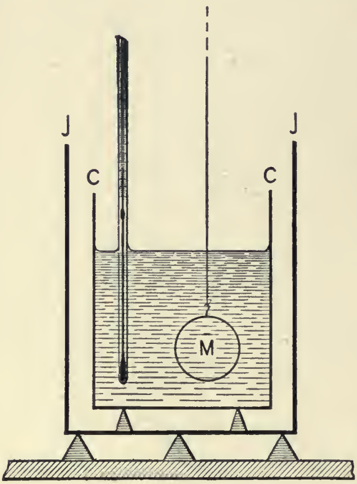

Fig. 127.

(*c*) In a vessel of water which is being heated, there are always large local differences of temperature. In order that the indications of the thermometer be accurate, brisk stirring is necessary.

238. The ice calorimeter. — To change one gram of ice at the freezing point into water at the same temperature 80 gram-calories of heat are required. An amount of heat may therefore be measured by determining the amount of ice it will melt. The amount of ice melted may be determined by weighing the ice water or by observing the decrease in volume.

The latter method is that employed in the ice calorimeter of Bunsen. This instrument, which is probably the most sensitive of all calorimetric devices, is shown in Fig. 128. It consists of a glass tube *T*, within which a smaller tube, in shape and size similar to the ordinary "test tube" of the chemist, is sealed. The latter receives the material *m*, which has previously been heated to a known temperature, and which is to be tested.

The upper part of (*T*) is completely filled with water from which all air has been removed. The lower part is filled with mercury which fills also the narrower tube (*t*) and a part of the capillary neck (*n*). Before the determination the calorimeter is packed in a bath of melting ice. The receiving tube is then filled with a freezing mixture by the action of which it becomes encased with ice. The increase of volume drives mercury from

(*T*) and produces a movement of the mercury column in (*n*). The freezing mixture having been removed from the receiving tube the calorimeter is kept in its ice bath until the end of the mercury column comes to rest. The heated substance is then introduced into the receiving tube. The heat which it imparts in cooling to the ice temperature is all employed in melting the surrounding bulb of ice. The decrease in volume, which is indicated with great precision by the retreat of the mercury column in the neck (*n*), affords a measure of the mass of ice which is melted, and so, indirectly of the thermal capacity of *m*.

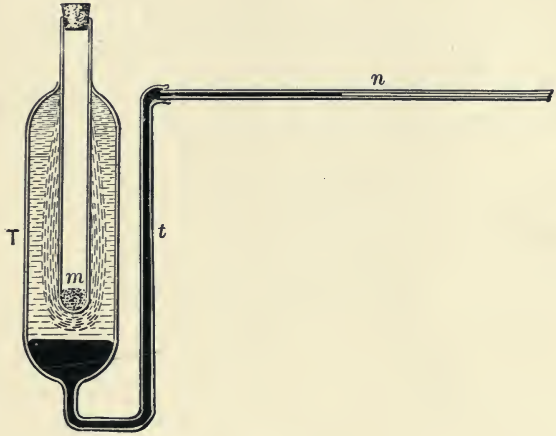

Fig. 128.

239. Specific heat. — The number of thermal units required to raise the temperature of unit mass of a substance 1 degree is called the specific heat of the substance.

TABLE V.*

Specific Heats of some of the Common Elements.

Substances.	Temps.	Sp. Heat.	Substances.	Temps.	Sp. Heat.
Aluminium . . .	0°–100°	0.2185	Manganese .	14°– 97°	0.1217
Antimony . . .	0°–100°	0.495	Mercury . .	20°– 50°	0.0331
Bismuth	20°– 84°	0.305	Nickel. . .	14°– 97°	0.1091
Cadmium . . .	0°–100°	0.1804	Phosphorus:		
Carbon (diamond)	0°–100°	0.145	(Red) . .	15°– 98°	0.1698
" (graphite)	0°–100°	0.186	(Yellow) .	13°– 36°	0.202
Copper	0°–100°	0.0933	Platinum . .	0°–100°	0.0323
Gold	0°–100°	0.0316	Silver . . .	0°–100°	0.0568
Iron	0°–100°	0.1130	Sulphur . .	15°– 97°	0.1844
Lead	0°–100°	0.0315	Tin	0°–100°	0.0559
Magnesium . . .	20°– 51°	0.245	Zinc . . .	0°–100°	0.0938

TABLE VI.

Specific Heats of Compounds.

Substances.	Temps.	Sp. Heat.	Substances.	Temps.	Sp. Heat.
Bell metal . . .	15°– 98°	0.085	Glass:		
Brass	0°	0.089	(Plate) .	10°– 50°	0.186
Bronze	20°–100°	0.1043	(Crown) .	10°– 50°	0.161
(88.7 cu. + 11.3 al.)			(Flint) . .	10°– 50°	0.117
German silver . .	0°–100°	0.0946	——		
——			Paraffin . .	10°– 15°	0.562
Quartz	20°– 50°	0.186	Wax . . .	26°– 42°	0.82
Granite	12°–100°	0.190	Vulcanite . .	20°–100°	0.331
Marble	18°– 99°	0.208	Ice	−30°– 0°	0.505

In the above table the temperatures between which the substances were tested, or to which the value of the specific heat applies, is given in each case. This is important because the *specific heat of most (probably of all) substances varies with the temperature.* The general character of the influence of tem-

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* For more complete data, see Landolt and Börnstein, *Physikalisch-chemische Tabellen.*

perature upon thermal capacity may be illustrated by reference to the case of carbon in the form of the diamond, a substance the specific heat of which has been studied at various temperatures between $-50°$ C. and $+1000°$ C., and of iron, which has been tested through almost as great a range. The changes are indicated graphically in Fig. 129 by means of a curve, the

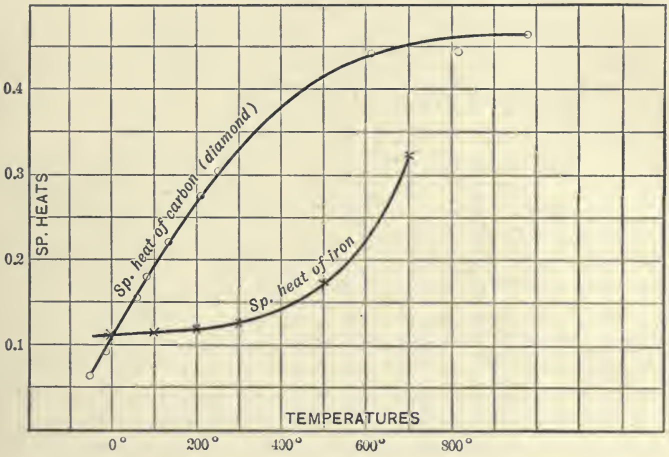

Fig. 129.

ordinates of which are specific heats, and the abscissas temperatures.

In the case of water, which is of special interest, because that substance is taken as a standard of reference in nearly all calometric determinations, the influence of temperature upon specific heat is much less marked than in the two examples just cited. That it is appreciable, however, will be seen from the following table, which gives the specific heat of water at various temperatures in terms of that at 15° as unity.

TABLE VII.

SPECIFIC HEAT OF WATER.

TEMPERATURE.	SP. HEAT.		TEMPERATURE.	SP. HEAT.	
0°	1.00664		16°	0.99983	
2°	1.00543		18°	0.99959	
4°	1.00435		20°	0.99947	
6°	1.00331		22°	0.99955	
8°	1.00233		24°	0.99983	
10°	1.00149		26°	1.00031	
12°	1.00078		28°	1.00098	
14°	1.00023		30°	1.00187	

240. Method of mixtures. — The method usually employed for the measurement of specific heat is known as the method of mixtures. The substance, weighing S grams, is heated to a temperature t, quickly plunged into a water calorimeter at temperature t', and the final temperature t'' of the mixture is observed. The heat given off by the S grams of substance in cooling from t to t'' is equal to the heat absorbed by the water of the calorimeter in being heated from t' to t''. This, by equation (146), is $W(t''-t')$. Therefore the specific heat of the substance is

$$\text{Sp. Heat} = \frac{W(t''-t')}{S(t-t'')}. \tag{147}$$

The specific heat is the number of thermal units given off by one gram of the substance cooling through one degree.

241. Heat of combustion. — Chemical action is in general accompanied by the generation or the extinction of heat. Those chemical reactions during the progress of which heat is generated are called *exothermic* reactions. Those reactions during the progress of which heat is absorbed or extinguished are called *endothermic* reactions. Combustion is the most important case of exothermic reaction. The heat generated per unit mass of a substance burned is called the heat of combustion of that substance.

The method of measuring heats of combustion is analogous to that for specific heat. On account of the very great volume occupied by the gaseous products of the chemical reaction, however, it is necessary to employ a special form of calorimeter. This usually consists of a metallic vessel *aa* (Fig. 130), containing a smaller one, *A*, within which the process of combustion is carried on, and a hollow spiral coil through which the products pass at such a rate that they will be cooled to the temperature of the water which surrounds both the coil and the combustion chamber.

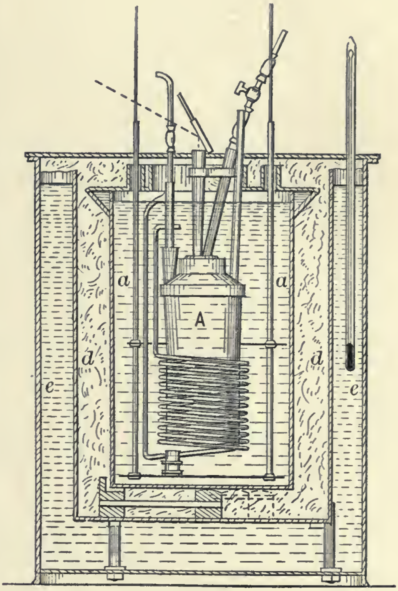

Fig. 130.

The calorimeter proper, *aa*, is surrounded by a system of double jackets, *dd*, containing swan's-down, and *ee*, a water jacket.

The accessory parts, by means of which complete combustion is secured, are not shown in the diagram. The heat of combustion of certain substances is given in the following table.

TABLE VIII.

HEAT OF COMBUSTION.

SUBSTANCE.	PRODUCT OF COMBUSTION.	HEAT OF COMBUSTION.	SUBSTANCE.	PRODUCT OF COMBUSTION.	HEAT OF COMBUSTION.
Carbon (diamond) .	CO^2	7770	Illuminating gas .	—	5200 to 6365
" (graphite) .	CO^2	7762 to 8137	Raw petroleum . .	—	11094
Hydrogen	H_2O	34700	Refined petroleum	—	11045
Magnesium . . .	MgO	6010	Gunpowder . . .	—	508 to 807
Phosphorus (yellow)	P_2O_5	5747	Gun cotton . . .	—	1056
Sulphur	SO_2	2220	Dynamite. . . .	—	1290
Various woods . .	—	4100 to 4500	Methane (CH_4) .	—	13063
Charcoals	—	7071 to 8080	Benzol (C_6H_6) . .	—	10000
Soft coals	—	7400 to 8800	Methyl alcohol . .	—	5307
Anthracite	—	7844	Ethyl alcohol . .	—	7183

242. Changes of state with rise of temperature. — When heat is imparted to a solid, the temperature rises until the solid begins to melt. The temperature then remains constant until all of the substance is changed into the liquid state, when it begins to rise again and continues to rise until the liquid boils. The temperature then remains constant until the liquid is entirely changed to vapor, when it once more begins to rise. There are thus two periods during which heat is imparted to the substance without producing rise of temperature. These are called respectively the *melting point* and the *boiling point.*

The melting point may be further defined as the temperature at which the solid and liquid forms of the substance are capable of existing together in equilibrium. This temperature varies but slightly with pressure. The variation may consist either in a lowering of the melting point with pressure (case of all bodies which like ice expand upon freezing), or it may consist in a raising of the melting point (case of bodies which expand upon melting).

The following table gives the melting points of some solids at 76 cm. pressure.

TABLE IX.

MELTING POINTS OF VARIOUS SOLIDS.

I. *Elements.*

SUBSTANCE.	TEMPERATURE.	SUBSTANCE.	TEMPERATURE.
Aluminium	600°–850° *	Mercury	−39°
Antimony	425°–450°	Lithium	180°
Bismuth	260°	Nickel	1476°–1517°
Bromine	7°.3	Nitrogen	−203°–214°
Cadmium	315°	Osmium	2500° (?)
Chlorine	−102°	Phosphorus	44°.3
Cobalt	1500°–1800°	Platinum	1757°–1780°
Copper	1050°–1100°	Potassium	62°
Gold	1030°–1250°	Selenium	217°
Iodine	113°.6	Silver	950°–1032°
Iridium	1950°–2000°	Sodium	90°
Iron	1200°–1800°	Sulphur	113°
Lead	325°	Tellurium	452°–525°
Magnesium	750°–800°	Tin	226°
Manganese	1900° (?)	Zinc	400°–420°
II. *Compounds, etc.*			
Ethylen (C_2H_4)	−169°	Methane (CH_4)	−185°
Ammonia (NH_3)	− 75°	Nitrous oxide (N_2O)	− 99°
Carbon monoxide (CO)	−207°	Sulphur dioxide (SO_2)	− 76°
Carbon dioxide (CO_2)	− 57°	Sulphuretted hydrogen (H_2S)	− 91°
Carbon disulphide (CS_2)	−110°		
Hydrochloric acid (HCl)	−112°		
III. *Alloys.*			
Lead 46.7%; Tin 53.3%	+197°	Wood's metal (Cd. 12.5%; Sn. 12.5%; Pb. 25%; Bi. 50%)	+ 70°
Britannia metal (Zn. 82%; Sb. 18%)	+250°		
Rose's metal (Sn. 25%; Pb. 25%; Bi. 50%)	+ 95°		

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* Where two values are given, authorities differ as to the melting point. This happens chiefly in the case of very high and very low temperatures.

243. Minimum temperature of a vapor at a given pressure and maximum pressure of a vapor at a given temperature. —(*Definition of boiling point.*) The vapor of a given substance under given pressure condenses to a liquid if an attempt is made to cool it below a certain minimum temperature. This temperature is called the *boiling point* of the liquid at the given pressure. The temperature at which a liquid boils at a given pressure is usually somewhat higher than its *true boiling point*, partly because of increased pressure in the liquid due to gravity, and partly because of the fact that vapor can only be formed at the surface of the liquid or at some surface of discontinuity in the liquid, *e.g.* on dust particles, etc.

The vapor of a given substance at given temperature condenses to a liquid if an attempt is made to increase its pressure, *e.g.* by compression, beyond a certain maximum value.

The relations between the pressure and temperature of water vapor and the vapor of ammonia respectively are given in the following tables, and graphically in Figs. 131 and 132.

TABLE X.

PRESSURE AND TEMPERATURE OF SATURATED AQUEOUS VAPOR.

(FROM − 10° C. TO + 230° C.).

TEMP.	PRESSURE IN CENTIMETERS.	TEMP.	PRESSURE IN CENTIMETERS.	TEMP.	PRESSURE IN CENTIMETERS.	TEMP.	PRESSURE IN CENTIMETERS.
− 10°C.	0.2151 cm.	55°	11.7516 cm.	101°C.	78.757 cm.	150°	358.123 cm.
− 5°	0.3160	60°	14.8885	102°	81.601	155°	408.856
0°	0.4569	65°	18.7103	103°	84.528	160°	465.162
+ 5°	0.6507	70°	23.3308	104°	87.541	165°	527.454
10°	0.6971	75°	28.8764	105°	90.641	170°	596.166
15°	1.2674	80°	35.4873	110°	107.537	175°	671.743
20°	1.7363	85°	43.3194	115°	126.941	180°	754.692
25°	2.3517	90°	52.5468	120°	149.128	185°	845.323
30°	3.1510	95°	63.3657	125°	174.388	190°	944.270
35°	4.1784	96°	65.7396	130°	203.028	195°	1051.963
40°	5.4865	97°	68.1879	135°	235.373	200°	1168.896
45°	7.1362	98°	70.7127	140°	271.763	210°	1432.480
50°	9.1978	99°	73.3160	145°	312.555	220°	1739.036
		100°	76.0000			230°	2092.640

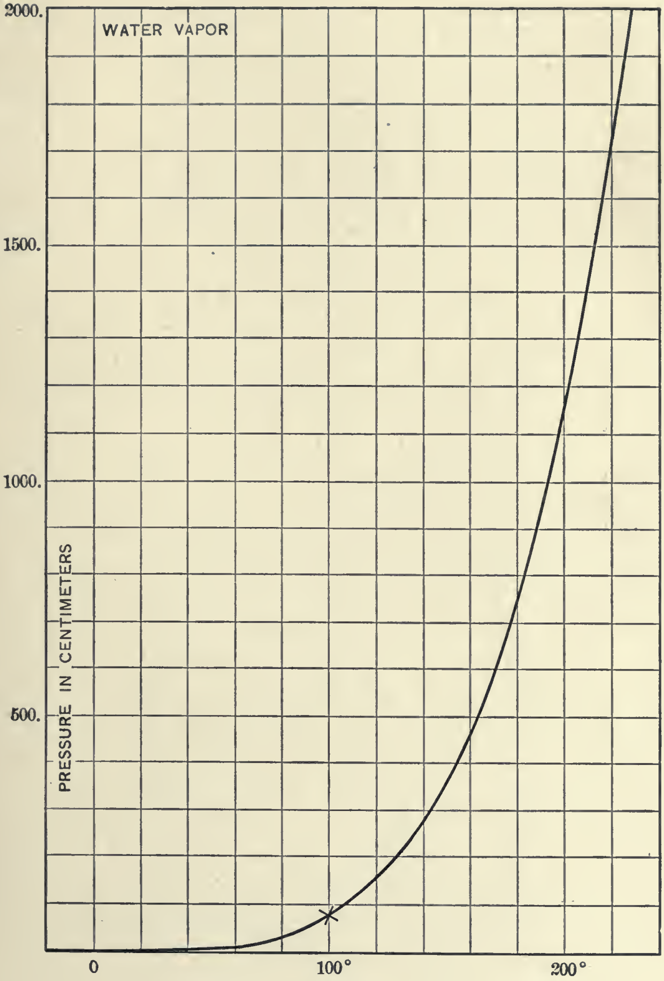
WATER VAPOR
PRESSURE IN CENTIMETERS
2000.
1500.
1000.
500.
0
100°
200°

Fig. 131.

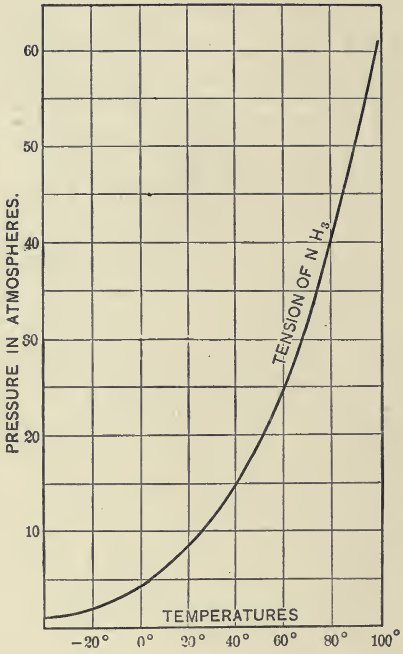

Fig. 132.

TABLE XI.

PRESSURE AND TEMPERATURE OF SATURATED VAPOR OF AMMONIA (NH_3). (REGNAULT).

TEMP.	PRESSURE IN ATMOSPHERES.	TEMP.	PRESSURE IN ATMOSPHERES.
− 30° C.	1.14 atm.	50°	19.95 atm.
− 20°	1.83	60°	25.63
− 10°	2.82	70°	32.47
0°	4.19	80°	40.59
10°	6.02	90°	50.14
20°	8.41	100°	61.32
30°	11.45		
40°	15.26		

The following are the boiling points of various liquids at 76 cm. pressure:

SUBSTANCE.	BOILING POINT.	SUBSTANCE.	BOILING POINT.
Bromine	63.° C.	Nitrous oxide (N_2O)	− 88.° 8
Chlorine	− 33.° 6	Nitric oxide (NO)	− 153.° 6
Mercury	357.° 25	Sulphuretted hydrogen (H_2S)	− 63.° 5
Nitrogen	− 193.°	Sulphur dioxide (SO_2)	− 10.°
Oxygen	− 184.°	Acetic acid	− 118.° 1
Phosphorus	285.°	Alcohol (ethyl)	78.° 40
Selenium	665.°	Alcohol (methyl)	66.°
Sulphur	448.° 4	Benzol	80.° 36
Zinc	929.°	Chloroform	51.° 20
Ethylen (C_2H_4)	− 102.°	Glycerine	290.°
Carbon disulphide (CS_2)	46.°04	Pentane (M)	37.°
Cyan (CN_2)	− 20.° 7		
Methane (CH_4)	− 164.°		

244. Heat of fusion and of vaporization. — When heat is applied to a substance which is melting or is in ebullition, the temperature remains constant because the heat-energy imparted to the substance is employed in producing the change of state.

The heat of fusion, according to the above statement, is the number of thermal units required to change unit mass of the solid at its melting point into liquid at the same temperature. That the heat of fusion, which is capable of definite and direct determination, is much greater in the case of water than with other substances, will be seen from the following table:

TABLE XII.

HEAT OF FUSION OF VARIOUS SUBSTANCES.

Bismuth	12.640 cal.	Mercury	2.82
Bromine	16.185	Paraffin	35.10
Cadmium	13.660	Phosphorus	4.744
Glycerine	42.50	Platinum	27.18
Ice	79.25	Silver	21.07
Iron	33–50	Sulphur	9.365
Lead	5.858	Tin	13.314

The heat of vaporization of a liquid is the number of thermal units required to change unit mass of that liquid at its boiling point into vapor at the same temperature. This quantity, also, is capable of direct measurement.

245. Determination of the heat of vaporization. — The apparatus employed in the measurement of the heat of vaporization is a calorimeter containing a hollow spiral and an inner receptacle. Vapor is sent into the spiral, where it is condensed, and the liquid thus produced gathers in the receiver, whence it is subsequently removed and weighed.

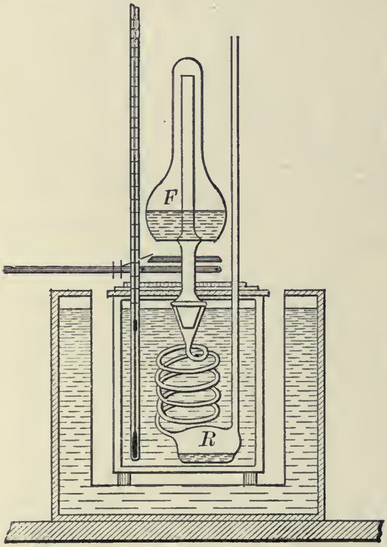

Fig. 133.

Fig. 133 shows the essential features of one of the best known forms of apparatus, that of Berthelot.

The liquid is heated in the vessel *F*, whence its vapor passes through the inner tube to the spiral. Here it condenses, warming the surrounding water of the calorimeter, and is collected in the receptacle *R*.

The following table gives the heat of vaporization of various liquids:

TABLE XIII.

Alcohol (ethyl) .	208.92 cal.	Ether (C_4H_{10}) .	91.11
Ammonia (NH_3) .	294.21 (at 7°.8)	Iodine	23.95
Benzol	93.45	Mercury . . .	62.00
Bromine	45.60	Sulphur dioxide .	91.7 (at 0°)
Carbon dioxide . .	56.25 (at 0°)	Water	535.9
Carbon disulphide .	86.67		
Chloroform . . .	58.49		

246. Relation between heat of vaporization and temperature.— The heat of vaporization depends upon the temperature at which ebullition takes place, diminishing rapidly as the temperature rises. In the case of each liquid it is supposed that the heat of vaporization would become zero at the highest temperature at which the substance in question is capable of existing in the liquid state. In point of fact it has been found by means of experiments upon liquids heated in closed tubes, that at a definite temperature which is constant for each liquid, the surface film disappears and it becomes impossible to distinguish between the liquid in the bottom of the tube and the saturated atmosphere of vapor above it. The temperature at which this change occurs is called the **critical temperature.*** The identity of the critical temperature as above defined and the temperature at which the heat of vaporization vanishes has not been experimentally established. It is, however, known that the heat of vaporization of such liquids as have been investigated diminishes rapidly throughout the entire range of temperatures to which the experiments have been extended. Thus the heat of vaporization of water is 606.5 at 0°, 535.9 at 100°, and 464.3 at 200°.

In the following table the critical temperatures of certain substances are given:

TABLE XIV.

CRITICAL TEMPERATURES OF VARIOUS LIQUIDS.

Alcohol (ethyl) . . .	240° C.	Methane (CH_4)	− 81° C.
Ammonia (NH_3) . .	130°	Nitrogen	− 146°
Benzol	280°	Nitrous oxide (N_2O) . . .	35°.4
Bromine	302°	Oxygen	− 118°
Carbon monoxide (CO)	141°	Sulphuretted hydrogen (H_2S)	100°
" dioxide (CO_2) .	31°	Sulphur dioxide (SO_2) . .	156°
Chlorine	141°	Turpentine oil	376°
Chloroform	260°	Water	358° (365°?)

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* For a description of the classical experiments of Andrews upon the critical temperature, see Philosophical Transactions 1869, II. p. 575; also Preston, Theory of Heat, p. 372.

247. Evaporation. — A liquid at a given temperature continues to evaporate so long as the *pressure of its vapor* is less than the maximum pressure at the given temperature. This is true, whether the space above the liquid is filled with vapor alone, or with vapor mixed with any gases at any pressure. See Arts. 248, 254.

248. Hygrometry. — The condition of the atmosphere in regard to the moisture it contains is characterized by the following quantities — not all independent.

"Force of vapor." The pressure of the atmosphere is due in part to each of its constituents. According to Dalton's Law (Art. 254), the pressure due to each constituent is equal to the pressure which would be exerted by that constituent if it occupied alone the whole space. That part of the atmospheric pressure which is due to water vapor is called in meteorology "Force of Vapor." This pressure varies from zero to 30 mm. or more.

The **dew point** is the temperature to which the atmosphere must be cooled in order that the water vapor present may be saturated. The atmosphere at the dew point is said to be saturated with moisture.

The **relative humidity** of the atmosphere is the amount of water vapor in the air expressed in hundredths of what the air would contain were it saturated at the given temperature.

The **absolute humidity** is the actual amount of water in the air, generally expressed in grams per cubic meter of air.

The experimental determination of these various quantities is called **Hygrometry.** The method most employed is by use of wet and dry bulb thermometers, from the readings of which the various quantities may be determined from empirical tables.

CHAPTER XI.

THE PROPERTIES OF GASES.*

249. Restatement of Gay Lussac's Law. — From Arts. 224, 225, and 227, we have:

(*a*) *The pressure of a constant volume of any gas is proportional to its absolute temperature.*†

This, together with Boyle's Law (Art. 167), gives:

(*b*) *The volume of a gas is proportional to the absolute temperature, the pressure being constant.*†

The laws of Boyle and Gay Lussac may be embodied in the following formula; viz.,

$$pv = RT, \tag{148}$$

in which p is the pressure of a gas, v its volume, T its absolute temperature, and R a proportionality factor depending upon the quantity and nature of the gas. This equation completely expresses both Boyle's and Gay Lussac's laws.

250. Another important law of gases which should be stated here is **the law of multiple volumes in the chemical combination of gases** (Berzelius); viz.: *The ratios of the volumes, reckoned at same temperature and pressure of two or more gases which enter into chemical combination, are simple and rational.* For example, one volume of hydrogen and one volume of chlorine combine to form HCl; two volumes of hydrogen and one volume

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* See Art. 167 *et seq.* for statement of Boyle's Law.

† These statements involve both the absolute statements given in Art. 224, and the provisional definition of temperature given in Art. 225. They are therefore slightly misleading inasmuch as they make it appear that temperature has been previously and independently defined.

of oxygen combine to form water; ammonia upon dissociation gives three volumes of hydrogen and one volume of nitrogen.

Other experimental facts concerning gases will be given in the course of the following development of the kinetic theory.

251. Kinetic theory of gases. Fundamental assumptions. — The laws of Boyle and Gay Lussac are satisfied if we assume a gas to consist of a large number of very small moving particles — molecules — which rebound with unchanged* velocity when they strike the walls of the containing vessel, which are so small as seldom to collide with each other, and which exert no perceptible mutual attraction.

The kinetic energy of such a system of particles is constant, therefore the *average square*, ω^2, of the velocities of the particles is constant and definite. Let p be the pressure of the gas in $\frac{\text{dynes}}{\text{q.c.m.}}$, v the volume of the containing vessel, N the total number of particles, $n\left(=\frac{N}{v}\right)$ the number of particles per cubic centimeter, and m the mass of each particle. Then we have

$$p = \tfrac{1}{3} nm\omega^2 \tag{149}$$

or

$$p = \tfrac{1}{3} \frac{Nm\omega^2}{v}. \tag{150}$$

Proof. — The square of the velocity of a given particle is equal to the sum of the squares of the x, y, and z components of its velocity. Therefore the sum of the squares of the velocities of all the particles is equal to the sum of the squares of all the x-components, *plus* the sum of the squares of all the y-components, *plus* the sum of the squares of all the z-components. The particles move at random in all directions, so that the sum of the squares of the x-components, of the y-components, and of

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* When the walls of a vessel are hotter than the gas, the molecules rebound with increased velocity, and *vice versa*.

the z-components are equal each to each. Therefore (a) *The sum of the squares, $N\omega^2$, of all the velocities is equal to three times the sum of the squares of the x-components.*

Imagine the containing vessel to consist of two parallel walls, of area q, distant d from each other, perpendicular to the x-axis of reference, and between which the gas is confined. Only the x-components of the molecular velocities contribute, by impact, to the pressure on these walls, so that the y and z components may be ignored. Consider a single particle, the x-component of whose velocity is a. This particle strikes first one wall and then the other, traveling back and forth $\frac{a}{2d}$ times per second. At each impact the velocity of the particle changes by $2a$, that is, from $+a$ to $-a$, or the momentum of the particle changes by $2am$. Therefore momentum is lost on each wall by the impact of this particle at the average rate $\frac{a}{2d} \times 2am$, or $\frac{m}{d}a^2$, which is the average force exerted on the wall by this particle. That is, the force on one wall, due to one particle, is equal to $\frac{m}{d}$ times the square of its x-velocity component. Therefore the total force F, exerted on the wall by all the particles, is equal to $\frac{m}{d}$ times the sum of the squares of their x-velocity components. Therefore $F = \frac{m}{d}\frac{1}{3}\omega^2 N$; see ($a$). Dividing by q, and putting $qd = v$, we have $\frac{F}{q} = p = \frac{1}{3}\frac{Nm\omega^2}{v}$.

252. Further assumption. — The absolute temperature of a gas is assumed to be proportional to $\frac{1}{2}m\omega^2$, *i.e.* to the average kinetic energy per molecule. In accordance with this assumption, we may write RT for $\frac{1}{3}Nm\omega^2$ in equation (150), R being a proportionality factor. Equation (150) then becomes $pv = RT$, which, being identical to equation (148), shows that a gas may be considered to be such a system of particles as has been described. The various assumptions are therefore justified.

253. Calculation of ω. — Solving equation (149) for ω, we have $\omega = \sqrt{\frac{3p}{nm}}$. The product nm is equal to the density of the gas, therefore easily observed, as is also p, so that ω may be calculated.

Example. — The density of hydrogen gas at 0° C., and under a pressure of 1,013,373 $\frac{\text{dynes}}{\text{q.c.m.}}$ (= 760 mm.) is 0.00008954 $\frac{\text{grams}}{\text{c.c.}}$. Therefore, at 0° C. the square root of mean square of the velocity of hydrogen molecules is 184,260 $\frac{\text{cm.}}{\text{sec.}}$.

254. Dalton's law. — *A mixture of gases, having no chemical action on each other, exerts a pressure which is equal to the sum of the pressures which would be produced by each gas separately, provided it occupied the containing vessel alone at the given temperature.* The kinetic theory is entirely consistent with this law, since the moving particles are assumed so small that they do not interfere with each other in any way.

255. Avogadro's principle. — The molecular theory, as devised to explain the law of multiple proportions in chemistry, together with Berzelius' law (Art. 250), indicates pretty clearly that *all gases contain the same number of molecules per cubic centimeter at the same temperature and pressure.* That this is consistent with the kinetic theory of gases, may be shown as follows: Consider two gases. Let p_1, n_1, m_1, and ω_1^2 be the pressure, number of particles per cubic centimeter, etc., of the one, and p_2, n_2, m_2, and ω_2^2 the corresponding quantities of the other gas. Then $p_1 = \frac{1}{3} n_1 m_1 \omega_1^2$, and $p_2 = \frac{1}{3} n_2 m_2 \omega_2^2$ from equation (149). If $p_1 = p_2$, and if the temperatures of the gases are equal, requiring $m_1\omega_1^2$ to be equal to $m_2\omega_2^2$ (see Art. 252), then $n_1 = n_2$.

256. Deviation of gases from Boyle's law. — All gases deviate more or less from Boyle's law. Such gases as carbon dioxide, alcohol vapor, etc., which have complex molecules, generally show the most marked deviation. Two, in particular, of the fundamental assumptions of the kinetic theory seem not to be strictly satisfied by gases, viz.: (*a*) That the particles are so small as seldom to collide. (*b*) That the particles exert no mutual attraction.

(*a*) **Effect of size of molecules.** — If the moving particles have any size, collisions and impacts take place before the centers of the particles are coincident, or before the centers of the particles are in the plane of the wall of the containing vessel. Shorter distances are thus traversed between collisions and impacts, which are therefore more frequent, and the pressure is greater than it would be if the particles were indefinitely small. The result is very much as if the volume of the containing vessel were smaller by a constant amount, b, than it really is. Equation (148) may be modified so as to take account of this deviation, by writing $v-b$ for v. The value of b depends upon the amount and nature of the gas, and its value for one gram of a gas is called the **molecular volume** of the gas.

(*b*) **The effect of mutual attraction of particles** is to slow up the particles as they come into the layers of the gas adjacent to the walls. The attraction of the walls is constant and need not be considered. This slowing up of the particles makes the pressure of the gas less than it would otherwise be, by an amount which can be shown to be proportional to the square of the density of the gas or inversely proportional to the square of its volume. Equation (148) may be modified, so as to take account of this deviation, by writing $p+\frac{a}{v^2}$ for p. The quantity a is a constant for a given amount of a given gas. Equation (148) therefore becomes

$$\left(p+\frac{a}{v^2}\right)\left(v-b\right)=RT. \qquad (151)$$

This equation is due to van der Waals.

257. Joule's and Thomson's experiment. — That the mutual attraction of the particles of a gas must be very small, was first shown by Joule. He placed a two-chambered vessel in a water calorimeter and found no perceptible rise or fall in temperature of the calorimeter when gas which had been compressed in one chamber was allowed to escape into the other, which was empty, by opening a cock. If the gas particles attract each other, they would lose some of their velocity as they move apart and the temperature would on the whole fall, and *vice versa*.

Joule and Thomson afterwards experimented with gases, allowing expansion to take place as the gas passed through a porous plug in a pipe. In case of molecular attraction, a cooling takes place as the gas passes the plug, and *vice versa*. Most gases showed slight cooling, and hydrogen alone was slightly warmed as it passed the plug. Hydrogen molecules at ordinary temperature and pressure, therefore, must be considered to repel each other slightly, so that for hydrogen the quantity a in equation (151) is negative.

258. A perfect gas. — A gas which follows Boyle's law strictly is called a perfect gas. The only energy in a perfect gas is the kinetic energy of its molecules. All gases whatever are *perfect* when highly rarefied. All simple gases, that is, gases of which the molecule is not complex, follow Boyle's law with a high degree of exactness even at ordinary temperatures and pressures. All the following calculations refer to perfect gases.

259. Clausius' ratio. — In the development of the kinetic theory of gases only *translatory* motion of the particles has been considered. All gas molecules are capable also of *rotatory* motion and *vibratory* motion. A part of the kinetic energy of a gas is due to translatory motion and is called *translational energy*, and part is due to rotatory and vibratory motion of the molecules and is called *intramolecular energy*. It was first pointed out by Clausius that the ratio of translational energy to intramolecular energy is constant for a given gas.

This ratio is called **Clausius' ratio.** The translational energy of a gas is $\frac{1}{2} Nm\omega^2$, the intramolecular energy is $\frac{1}{2}, \sigma Nm\omega^2$, and the total energy is $\frac{1}{2}(1+\sigma)Nm\omega^2$, in which σ is Clausius' ratio.

260. Two ways of imparting energy to a gas.

(*a*) Energy may be imparted to or extracted from a gas as heat by allowing the gas to come into contact with a hotter or colder body. Throughout this chapter heat is understood to be expressed in ergs.

(*b*) Energy may be imparted to or extracted from a gas as the work done upon the gas in compressing it or by the gas as it expands. Let ΔW be the work done upon a gas at pressure p during a decrement of volume Δv, then

$$\Delta W = -p \cdot \Delta v. \tag{152}$$

The negative sign is chosen for the reason that Δv is here negative, being a decrement, and it is desired to consider ΔW as positive when it represents work *done upon* the gas.

261. The two bulk moduli of a gas. — Let Δv be the decrement of volume of a gas produced by an increment of pressure Δp, v being the initial volume. Then $\frac{\Delta v}{v}$ is the isotropic strain produced by the stress Δp, and $\Delta p \Big/ \frac{\Delta v}{v}$ is the bulk modulus of the gas. If no heat is allowed to escape from the gas, the work done upon the gas during compression causes a rise in temperature. Therefore the decrement of volume, Δv, is less if no heat is allowed to escape than it is if the gas is compressed slowly and the heat allowed to escape so as to prevent a rise in temperature. The bulk modulus of a gas has therefore two important values.

(*a*) The modulus, M_e, corresponding to compression without loss of heat, called the **isentropic*** **modulus**.

(*b*) The modulus, M_t, corresponding to compression without change of temperature, called the **isothermic modulus**.

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* For the reason that what is called the *entropy* of the gas remains unchanged during such compression.

All substances have these two-bulk moduli, but it is only for gases that they are perceptibly different in value.

262. *Proposition.* — The ratio $\frac{M_e}{M_t}$ is equal to $\frac{5+3\sigma}{3+3\sigma}$, in which σ is Clausius' ratio. That is,

$$\frac{M_e}{M_t}=k, \tag{153}$$

where
$$k=\frac{5+3\sigma}{3+3\sigma}. \tag{154}$$

Proof. — (*a*) *Isothermic modulus.* — At constant temperature $Nm\omega^2$ is constant, therefore differentiating equation (150) we have $\Delta p=-\frac{1}{3}\frac{Nm\omega^2}{v^2}\Delta v$; writing in this expression p for $\frac{1}{3}\frac{Nm\omega^2}{v}$ and solving for

$$\frac{\Delta p}{\frac{\Delta v}{v}}\;(=M_t),$$

we have
$$M_t=-p. \tag{155}$$

(*b*) *Isentropic modulus.* — During the decrement of volume, work equal to $-p\Delta v$ is done upon the gas. This work goes to increase the total kinetic energy, $\frac{1}{2}(1+\sigma)Nm\omega^2$, of the gas. This increase in kinetic energy depends upon an increase in ω, since σ, N, and m are constant. Therefore

$$-p\Delta v=(1+\sigma)Nm\omega\cdot\Delta\omega. \tag{156}$$

The increase in pressure accompanying this compression is in part due to the decrement of volume and in part due to the increment of ω. Differentiating equation (150) first with respect to ω and then with respect to v, we have

$$\Delta p=\frac{2}{3}\frac{Nm\omega\cdot\Delta\omega}{v}-\frac{1}{3}\frac{Nm\omega^2\cdot\Delta v}{v^2}. \tag{157}$$

Substituting the value of $Nm\omega . \Delta\omega$ from (156) and the value $p=\frac{1}{3}\frac{Nm\omega^2}{v}$ from (150) in (157) and solving the resulting equation for $\Delta p \Big/ \frac{\Delta v}{v}$ $(=M_e)$, we have

$$M_e = -\frac{5+3\sigma}{3+3\sigma} \cdot p. \tag{158}$$

From (155) and (158) we have at once $M_e/M_t = k$.

263. Application of the above principles to the question of the velocity of sound. — Newton showed that the velocity, V, of sound in a substance is

$$V=\sqrt{\frac{M_e}{\rho}}, \tag{159}$$

in which ρ is the density of the substance. Newton calculated the velocity of sound in air from this equation, using the isothermic modulus. The velocity thus found is much less, in the ratio of $1 : \sqrt{\kappa}$, than the velocity given by observation. LaPlace was the first to point out that the isentropic modulus must be used in equation (159), since in a sound wave the changes of volume take place so rapidly that there is no perceptible loss of heat from the compressed portions of the air.

264. Experimental determination of M_e, M_t, σ, and κ **of a gas.** — From equation (155) M_t is shown to be equal to the *pressure* of the gas, and therefore easily observed.

Equation (159) enables the calculation of M_e when V and ρ have been observed.

Equation (153) enables the calculation of κ when M_t and M_e have been determined.

Equation (154) enables the calculation of σ when κ has been determined.*

Values of σ and κ have been determined in this manner for most gases. [See Ostwald, "Allgemeine Chemie," Bd. I., p. 248.] It is remarkable that $\sigma=0$ and $\kappa=\frac{5}{3}$, for monatomic

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* See Art. 281 for Clement and Desorme's method for the determination of κ.

gases, such as mercury vapor and argon. In such gases all the energy seems to be translational.

265. Specific heats of gases. — The number of thermal units required to raise the temperature of one gram of a gas one degree is called the specific heat of the gas. See Art. 239.

If the volume is maintained constant during this rise in temperature, since no external work is done by the gas, all of the heat applied goes to increase the total kinetic energy of the gas. If, however, the gas be allowed to expand, to such extent, for example, as to maintain the pressure constant, then additional heat will be required to make up for the work done by the gas during expansion.

The specific heat of a gas, therefore, has two important values.

(*a*) The specific heat, C_v, at constant volume.

(*b*) The specific heat, C_p, at constant pressure.

The latter has the larger value.

266. Proposition. — *The ratio of the two specific heats of a gas is equal to the ratio of the two bulk moduli.* That is,

$$\frac{M_e}{M_t}=\frac{C_p}{C_v}=\kappa. \tag{160}$$

Proof. — (*a*) *Preliminary statement.* C_p and C_v are proportional to the respective amounts of heat required to produce an arbitrary increment of temperature, ΔT, in an arbitrary amount, say N, molecules of the gas under the conditions of constant pressure and constant volume, respectively. Since T is proportional to the translational energy $\frac{1}{2}Nm\omega^2$ of the gas, the increment ΔT corresponds to a definite increment $Nm\omega\cdot\Delta\omega$ of this translational energy, or a definite increment of ω.

(*b*) *Heat required to raise temperature at constant volume.* In this case all the heat ΔH is used to increase the total kinetic energy, so that

$$\Delta H=(1+\sigma)\,Nm\omega\cdot\Delta\omega. \tag{a}$$

(*c*) *Heat required to raise temperature at constant pressure.* The heat, $\Delta H'$, required in this case exceeds ΔH by the amount $p \cdot \Delta v$, which amount is used for external work.

But $pv = \frac{1}{3} Nm\omega^2$ (from 150). Therefore since p is constant, $p \cdot \Delta v = \frac{2}{3} Nm\omega \cdot \Delta\omega$, so that

$$\Delta H' = (\tfrac{5}{3} + \sigma) Nm\omega \cdot \Delta\omega. \qquad (\beta)$$

Dividing (β) by (α) we have $\frac{C_p}{C_v} = \frac{\Delta H'}{\Delta H} = \kappa.$

267. Experimental determination of C_p. — In the method generally employed, the hot gas is drawn by means of an aspirator (pressure of cooling gas practically constant) through a coil of pipe in a water calorimeter. The mass of gas is determined from its density and the volume aspirated. The cooling of the gas is determined by observing temperatures at entrance to and exit from the coil, and the amount of heat in water units is determined by the rise in temperature of the calorimeter in the ordinary manner. This gives C_p in water units of heat per gram of gas per degree.

In Art. 282 a method is described for determining C_p in *ergs* per gram per degree. The ratio of the values of C_p determined in these two ways gives the number of ergs in one water unit of heat. This method for determining the mechanical equivalent of the water unit is due to Meyer.

The direct determination of C_v, by means of the calorimeter is scarcely possible, since the containing vessel used to maintain the volume of the gas constant must be heated and cooled with the gas, and the amount of heat required in thus heating and cooling the vessel is always many times as great as the heat given off by the cooling gas. The latter quantity is therefore largely masked by the experimental errors sure to occur.

Having determined C_p, however, and κ being known, C_v may be calculated from (160).

268. Constancy of Clausius' ratio. — From (α) and (β) of Art. 266 we see that C_v and C_p are proportional to $(1+\sigma)$ and to $(\frac{5}{3}+\sigma)$ respectively. C_v and C_p are shown by experiment to be constant for a given gas, therefore the constancy of σ is verified. Clausius showed, by considering the average behavior of a large number of erratically moving particles, that the ratio of their translational to their rotational and vibrational energy must be constant.

The experimental verification of this relation adds greatly to the strength of the kinetic theory of gases.

CHAPTER XII.

THERMODYNAMICS; THERMAL PROPERTIES OF HOMOGENEOUS SUBSTANCES.

269. Specification of physical state. — The physical state of a *given amount* of a gas, or of any homogeneous substance, is completely specified when its *pressure* and *volume* are given. Throughout this chapter *volume* will be understood to mean the volume of a definite amount of the substance (in general the volume of a gram of the substance). [See Art. 219.]

270. Watt's diagram. — It is convenient to represent the physical state of a gas (or of a homogeneous substance) by a point P (Fig. 134), of which the abscissa represents the volume, and the ordinate the pressure of the gas. The work $p\Delta v$ done by the gas in expanding is represented by the area $p\Delta v$ in the figure.

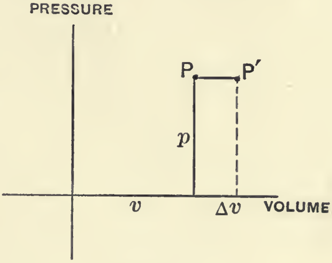

Fig. 134.

271. Process. — A substance which passes slowly from one physical state to another because of changing pressure or changing volume, or both, is said to undergo a *process*.

The points in Watt's diagram which represent the successive states passed through by a substance during a process, form a continuous curve. See Arts. 275 and 276. Thus a substance in passing from the state represented by the point P (Fig. 135) to the state represented by the point P' must pass without discontinuity through all the states represented by points lying on some continuous line connecting P and P'. The external work done in such a process is represented by the area $PP'ab$. This area is swept by the ordinate Pa as P moves along the curve to P'. It is customary to consider the work $p\Delta v$ as positive when Δv is negative, that is, when Δv is a decrement. Therefore the work represented by the area $PP'ab$ is positive when this area is swept to the *left* by the moving ordinate, and *vice versa*.

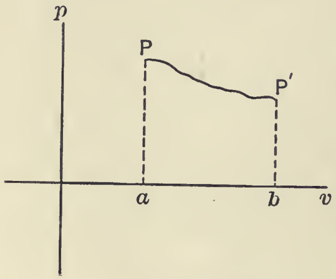

Fig. 135.

272. Cyclic process. — A process is said to be cyclic when the substance comes back at the end of the process to its initial state. Thus the closed curve (Fig. 136) represents a cyclic process. The substance, starting from the state P, passes through the states represented by the successive points on the line, and returns to the initial volume and pressure. If the process is performed in the sense indicated by the arrow, the shaded area will be swept once to the right and once to the left, while the enclosed area is swept once only and to the left. Therefore the enclosed area represents the work done upon the substance during the process. If the process is performed in the opposite sense, the enclosed area will represent the external work done *by* the substance during the process.

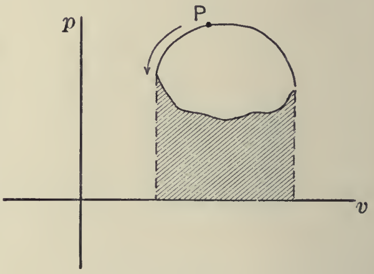

Fig. 136.

273. States* of thermal equilibrium. — A gas, or any homogeneous substance, if left standing, soon reaches a state in which

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* The word *state* implies *standing* or *permanence*, and there can be no such thing as a *state* other than equilibrium. The word is used, however, in speaking of *changing states*, that is, instantaneous aspects, of a system not in equilibrium.

the molecular velocity is the same in all its parts.* Such a state is called a state of thermal equilibrium of the substance. If a portion of the walls of the containing vessel is moved or heated, the molecules rebound from that portion with changed velocity. If the motion is very slow or the heating very slight, the state of the gas will be at each instant sensibly a state of thermal equilibrium. If the motion is rapid or the heating considerable, the distribution of molecular velocity will become distinctly non-uniform and the state of the gas distinctly other than a state of equilibrium.

274. Reversible processes. — If a material system be carried slowly through a series of states of equilibrium by weak external forces, it will be made to pass through this series of states in a reverse order if the external forces be reversed. In a similar manner, if a homogeneous substance be made to perform slowly any process by the action of external influences, the state of the substance will be at each instant a state of equilibrium and the process will be performed in a reverse order if the external influences be reversed. Such a process is called a reversible process.

Reversible processes are brought about only by the action of external influences on a substance under conditions such that the substance is at each instant in a state of equilibrium. The slow heating or cooling of a gas in a vessel is a reversible process. The slow compression or expansion of a gas in a cylinder under a piston is a reversible process (compare Art. 273).

275. Proposition. — The points in Watt's diagram which represent the successive states of a substance as it is carried through a reversible process form a continuous line. For the volume and pressure, and of course also temperature, must vary *slowly* in order that the substance may be at each instant in a

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* These parts are large compared to distance between molecules.

state of equilibrium and the process reversible (compare Arts. 273 and 274).

276. Sweeping processes. — A material system not in a state of equilibrium undergoes a *sweeping* change independently of the action of external force. After the system has settled to a state of equilibrium it could be made to pass through the process in a reverse direction only by bringing to bear upon it a system of external forces which exactly duplicates, with change of sign at each stage, the mutual forces which acted at the corresponding stage of the *sweep*.

A homogeneous substance not in a state of thermal equilibrium undergoes a *sweeping process* as the substance settles down to a state of thermal equilibrium. Such a process is *absolutely irreversible*, for no system of external forces can be devised duplicating the forces which act between the molecules of the substance during such a sweep. Sweeping processes are essentially independent of external influences. A gas confined to half of a closed vessel by a diaphragm undergoes a sweeping process when an aperture is opened in the diaphragm. A gas in a vessel which is quickly heated or cooled or of which the walls are moved rapidly, undergoes a sweeping process while the altered molecular velocity is being distributed throughout the gas. Most processes in nature are of a sweeping character and irreversible.

It is to be noticed that a gas, or any homogeneous substance, not in a state of thermal equilibrium, has no definite pressure, so that states of thermal equilibrium only, and reversible processes only, can be represented in Watt's diagram. If a substance passes from one state of thermal equilibrium to another by a sweeping process, this leaves a gap in the process-curve in Watt's diagram.

277. Isothermic and isentropic expansion. — Two reversible processes, which will be of particular importance when we come to the discussion of the second law of thermodynamics, are (a) expansion at constant temperature, that is *isothermic* expansion, and (b) expansion without gain or loss of heat, that is *isentropic* or *adiabatic* expansion. A line in Watt's diagram which shows the relation between p and v of a substance during isothermic expansion is called an **isothermic line**; a line which shows the relation between p and v of a substance during isentropic expansion is called an **isentropic** or **adiabatic** line.

278. Isothermic lines of gases. — When T is constant, equation (148) gives

$$pv = \text{constant}, \tag{161}$$

which is the equation to the isothermic lines of a gas, in Watt's diagram. These lines are evidently equilateral hyperbolas. For an example, see the continuous curves in Fig. 137.

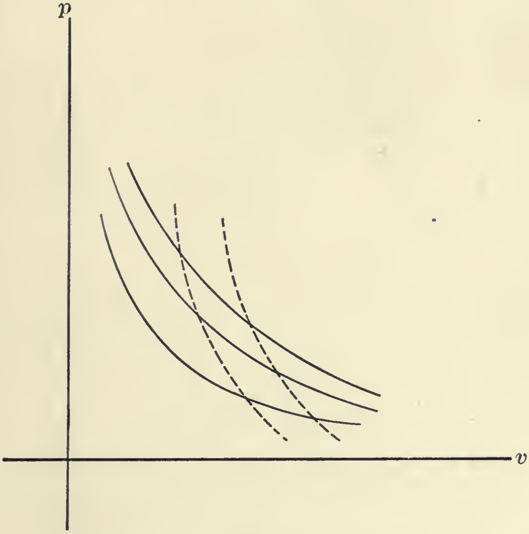

Fig. 137.

279. Isentropic lines of gases. — The equation to these lines is derived as follows: The external work $-p \cdot \Delta v$ done by a gas expanding without gain or loss of heat is equal to the decrement of the total kinetic energy, $\frac{1}{2}(1 + \sigma) Nm\omega^2$, of the gas. That is,

$$-p \cdot \Delta v = (1 + \sigma) Nm\omega \cdot \Delta\omega. \tag{a}$$

The change in pressure is in part due to the increase in volume and in part to the decrease of temperature, or of ω. Differentiating $p = \frac{1}{3}\frac{Nm\omega^2}{v}$ with respect to ω, and then with respect to v, we have

$$\Delta p = \frac{2}{3}\frac{Nm\omega \cdot \Delta\omega}{v} - \frac{1}{3}\frac{Nm\omega^2 \cdot \Delta v}{v^2}. \tag{b}$$

Substituting the value of $Nm\omega \cdot \Delta\omega$ from (a) and $p = \frac{1}{3}\frac{Nm\omega^2}{v}$ in (b), writing k for $\frac{5 + 3\sigma}{3 + 3\sigma}$ and reducing, we have

$$\frac{dp}{p} + \kappa\frac{dv}{v} = 0. \tag{c}$$

Integrating, we have

$$\log p + \kappa \log v = \text{constant} \tag{162}$$

or

$$pv^\kappa = \text{constant}, \tag{163}$$

which is the equation to the isentropic lines of a gas in Watt's diagram as shown by the dotted lines in Fig. 137.

280. Rise in temperature of a gas during isentropic compression. — Substituting the value of p from $pv = RT$ (148) in (163), we have

$$Tv^{\kappa - 1} = \text{constant}. \tag{164}$$

Substituting the value of v from $pv = RT$ (148) in (163) we have

$$Tp^{\frac{1-\kappa}{\kappa}} = \text{constant}. \tag{165}$$

If a gas at temperature T and volume v_1 be quickly compressed to volume v_2, then, since quick compression is isentropic, the temperature T_2 of the gas after compression will be such that

$$T_1 v_1^{\kappa-1} = T_2 v_2^{\kappa-1} \tag{a}$$

from equation (164). Similarly

$$T_1 p_1^{\frac{1-\kappa}{\kappa}} = T_2 p_2^{\frac{1-\kappa}{\kappa}}, \tag{b}$$

from equation (165), enables the calculation of the temperature T_2 to which a gas is raised, when, starting from temperature T_1 and pressure p_1, it is quickly compressed until its pressure is p_2.

281. Method of Clement and Desormes for the experimental determination of κ. — A vessel filled with the gas at pressure p_1 and temperature T_1 is opened to the air, allowing the pressure to drop quickly to atmospheric pressure p_2, the temperature falling at the same time to T_2. Equation (165) gives

$$T_1 p_1^{\frac{1-\kappa}{\kappa}} = T_2 p_2^{\frac{1-\kappa}{\kappa}}. \tag{a}$$

The vessel is then quickly closed and allowed to stand until it has reached its former temperature T_1, when the pressure is p_3, so that from Gay Lussac's law

$$\frac{T_2}{p_2} = \frac{T_1}{p_3}. \tag{b}$$

Substituting the value of T_2 from (*b*) in (*a*) and reducing, we have

$$\left(\frac{p_1}{p_2}\right)^{\frac{1-\kappa}{\kappa}} = \frac{p_2}{p_3}, \tag{166}$$

from which p_1, p_2, and p_3 having been observed, κ may be calculated.

282. Calculation of C_v and C_p from the data of the method of Clement and Desormes. — The external work, $\int p\,dv$, done by that part of the gas which after venting fills the vessel, during its isentropic expansion from p_1 to p_2, comes from the decrease in the energy of that part of the gas, and this decrease in energy it is which cools the gas from T_1 to T_2. Therefore, $\int pdv$ is equal to the heat given off by the part of the gas under consideration in cooling at *constant volume* from T_1 to T_2. Let ρ_2 be the density of the gas at pressure p_2 and temperature T_2, and v_2 the volume of the vessel. Then $\rho_2 v_2$ is the mass of the part of the gas under consideration, and $\rho_2 v_2(T_1 - T_2)C_v$ is the heat this amount of the gas would give off in cooling at constant volume from T_1 to T_2. Therefore

$$\int_{p_1}^{p_2} pdv = \rho_2 v_2 (T_1 - T_2) C_v. \qquad (a)^*$$

Upon reduction this gives:

$$C_v = \frac{p_2}{(\kappa - 1)\rho_2(T_1 - T_2)}\left[\left(\frac{p_1}{p_2}\right)^{\frac{\kappa-1}{\kappa}} - 1\right] \cdots \qquad (167)$$

The temperature T_1, not necessarily observed for the determination of κ, must be observed for the determination of C_v. The temperature T_2 cannot be observed directly but is to be calculated from equation (b) of Art. 281. Equation (167) then enables the calculation of C_v. Having thus determined C_v and having computed κ from (166), equation (160) enables the calculation of C_p.

283. Second law of thermodynamics. — *Heat cannot pass of itself from a cold to a hot body, nor can it be transferred by any means from a cold to a hot body without compensation.*

The transference of heat from a hot to a cold body is a sweeping process and is irreversible. This law may be stated by declaring the absolute irreversibility of any sweeping process. Thus, a gas cannot pass of itself from a region of low pressure

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[*] The integral $\int_{p_1}^{p_2} pdv$ is reduced as follows: The expansion being isentropic, we have $pv^\kappa = p_2 v_2^\kappa$ from equation (163). Substitute the value of dv from this expression, remembering that p_2 and v_2 are constants in the integral, and the integration becomes possible.

to a region of high pressure, nor can it be made to do so by any means without compensation.

The first part of this law, in either statement, is a matter of common experience. As to the second part, it *is* possible to transfer heat from a cold to a hot body, and to transfer a gas from a region of low pressure to a region of high pressure, but such transference depends in every case upon the doing of a definite amount of work or upon some outstanding change of some kind in some other substance or system. The doing of this work or this outstanding change is the *compensation* contemplated in the law. The strict equivalence of these two statements of the second law will appear when the principle of the equivalence of irreversible processes is fairly stated.

284. Carnot's cycle. — Let a portion of any homogeneous substance under a piston in a cylinder start from the state P (Fig. 138), the temperature being T_1, and pass through the following four processes, coming back to the initial volume and pressure as shown in the diagram. The diagram is constructed for a perfect gas.

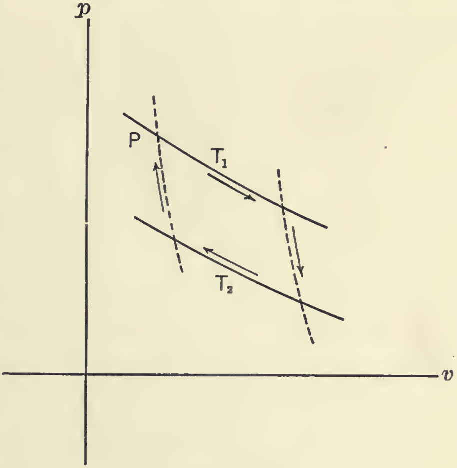

Fig. 138.

Let the gas (*a*) expand isothermally at temperature T_1.

(*b*) Expand isentropically from temperature T_1 to T_2.

(*c*) Be compressed isothermally at temperature T_2.

(*d*) Be compressed isentropically from temperature T_2 to T_1.

During the process (*a*) heat must be furnished to the substance to prevent its temperature falling, and during the process (*c*) heat must be extracted from the substance to prevent its temperature rising. During the processes (b) and (d) the substance must be in some manner isolated so as to be in contact with no substance with which it can exchange heat.

285. Carnot's apparatus. — The following ideal apparatus, hereafter called Carnot's apparatus, for brevity, serves to fix in the reader's mind the conditions of the performance of Carnot's cycle.

CC (Fig. 139) is a cylinder with piston PP, made of non-conducting material, the bottom of the cylinder only being a conductor.

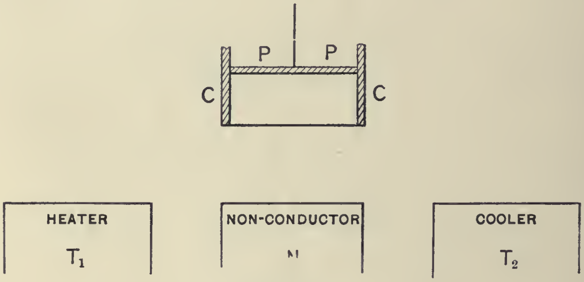

Fig. 139.

During the process (a) the cylinder is placed upon the *heater* at temperature T_1. As the substance slowly expands, the piston being slowly raised, heat from the heater passes through the conducting bottom and maintains the temperature of the substance constant. During the process (c) the cylinder is placed upon the cooler, which keeps the temperature constantly at T_2 during the compression. During the processes (b) and (d) the cylinder is placed upon the non-conducting table N and the substance thus thermally isolated.

286. Work gained and heat lost in Carnot's cycle. — When the substance has passed through Carnot's cycle, and has reached its initial state, its energy is as at first. Therefore, from the first law of thermodynamics,

$$W = H_1 - H_2, \tag{168}$$

in which W is the external work done *by* the substance during the cycle, H_1 is the heat absorbed from the heater, and H_2 is the heat given up to the cooler. The work W is represented by the enclosed area in Fig. 138 (see Art. 272).

287. Work spent and heat gained in Carnot's cycle reversed. — Carnot's cycle being carried out in a reversed direction, as is possible, since each process is reversible, the work W will be done upon the substance, the heat H_1 will be given to the heater, and the heat H_2 will be absorbed from the cooler.

288. Heat engine; refrigerating machine. — Carnot's apparatus is a **heat engine**, that is, a machine for transforming heat into work. The heat H_1 is taken from the heater; a portion of it is changed to work, and the remainder H_2 is given up to the cooler.

Carnot's apparatus driven backwards is a **refrigerating machine**, that is, a machine which extracts the heat H_2 from the cooler, and gives up this heat together with the heat equivalent of the work done on the substance, to the heater.

289. Efficiency of heat engines. — The ratio of the total heat H_1, extracted from the heater to the work W done by an engine, is called the efficiency η of the engine. That is,

$$\eta = \frac{W}{H_1}. \tag{169}$$

The ratio of the heat H_1 taken from the heater to the heat H_2 given to the cooler is also an important ratio. From equations (168) and (169) we have

$$\frac{H_2}{H_1} = 1 - \eta. \tag{170}$$

290. Proposition. — (*a*) No engine can be more efficient than Carnot's apparatus. (*b*) All reversible engines have the same efficiency as Carnot's apparatus. (*c*) Irreversible engines are less efficient than Carnot's apparatus.

Proof.—(*a*) Suppose there is an engine more efficient than Carnot's apparatus. (That is, doing the same work, but extracting less heat from the heater, and giving up less heat to the cooler than Carnot's apparatus.) Doing the same work, this engine will be able to drive Carnot's apparatus backwards. Carnot's apparatus will then extract more heat from the cooler than is given to it by the other engine, and will give off more heat to the heater than is taken from it by the other engine. Thus the arrangement will continue to transfer, on the whole, heat from the cooler to the heater, without compensation of any kind, which is contrary * to the second law. Therefore no engine can be more efficient than Carnot's apparatus.

(*b*) Suppose there is a reversible engine which is less efficient than Carnot's apparatus. (That is, doing the same work, but extracting more heat from the heater and giving more heat to the cooler.) This engine, driven backwards by Carnot's apparatus, will constitute an arrangement which will continue to transfer heat from cooler to heater without compensation. This is contrary to the second law; therefore no reversible engine can be less efficient than Carnot's apparatus.

(*c*) If any *irreversible* engine were of the same efficiency as Carnot's apparatus, this engine could drive Carnot's apparatus backwards, or be driven backwards by Carnot's apparatus, without transferring heat from heater to cooler, or *vice versa*. The arrangement thus being reversible, the engine itself must be so. Therefore no irreversible engine can be as efficient as Carnot's apparatus.

A reversible engine is called a *perfect* engine. All perfect

---

* The second part of the second law of thermodynamics, being largely outside of the reader's experience, will seem to him to afford an unsatisfactory basis of proof. In fact, the experimental verification of this second law comes largely from the experimental verification of the deductions made from it, just as in case of Newton's laws of motion.

engines are equally efficient when working between the same heater and cooler.

291. Rigorous definition of the ratio of two temperatures (Thomson). — The provisional definition of the ratio of two temperatures, given in Art. 225, leads to inconsistent results, unless a certain gas at a certain density is always used as the thermometric substance. (Because Gay Lussac's and Boyle's laws are only approximately true.) Lord Kelvin (Sir W. Thomson) has pointed out the following definition of temperature ratios, which is entirely independent of the thermal properties of any particular substance. All perfect engines have the same efficiency when working between the same temperatures. Therefore the efficiency of a perfect engine depends only upon the temperatures T_1 and T_2 of the heater and cooler, or, in other words, the ratio $\frac{H_1}{H_2}$ of the heat taken from the heater to the heat given to the cooler, by a perfect engine, depends only upon the temperatures T_1 and T_2. Therefore the ratio of the two temperatures T_1 and T_2 may be defined as the ratio of the heats H_1 and H_2. That is,

$$\frac{T_1}{T_2}=\frac{H_1}{H_2}, \tag{171}$$

and *the ratio of two temperatures is equal to the ratio of the heats extracted from a heater and given to a cooler at the respective temperatures, by a perfect thermodynamic engine working between said heater and cooler.*

292. Transformation of equation (171). — It is convenient to consider heat as *positive* when absorbed by a substance which is undergoing a process which is the object of consideration, and *vice versa.* Therefore H_1 in equation (171) is positive, and H_2 is negative, and this equation may be written $\frac{T_1}{T_2}=-\frac{H_1}{H_2}$, or more conveniently:

$$\frac{H_1}{T_1}+\frac{H_2}{T_2}=0. \tag{172}$$

293. Proposition. — The sum of the quotients, $\frac{\Delta H}{T}$, of the heat absorbed (or given off) by a substance during each step of a cyclic process, divided by the absolute temperature at the step, is zero; provided that no part of the cycle is of a sweeping character.

For a cyclic process, therefore,

$$\Sigma \frac{\Delta H}{T} = 0. \tag{173}$$

Proof. — Let the closed line CD (Fig. 140) represent a cyclic process. Draw two isentropic lines e and f *very* close together. The portions of CD which fall between e and f may be considered to be portions of isothermals* a and b, corresponding to temperatures T_1 and T_2 respectively.

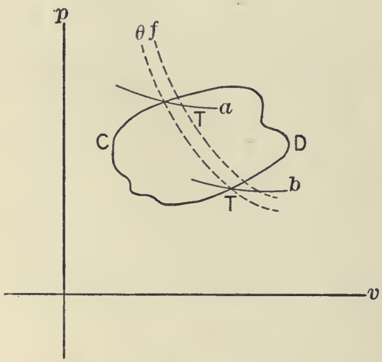

Fig. 140.

The two isentropic lines e and f, together with the intervening portions of the curve CD, thus form a Carnot's cycle. Let ΔH_1 and ΔH_2 be the heats absorbed at temperatures T_1 and T_2 respectively. Then, from equation (172),

$$\frac{\Delta H_1}{T_1} + \frac{\Delta H_2}{T_2} = 0.$$

The given cycle CD may be broken up in this manner into a large number of Carnot's cycles, so that for each portion of heat ΔH_1 absorbed by the substance during the cycle, there is another portion ΔH_2, which together satisfy equation (172). Therefore equation (173) is true.

---

* These short portions of C and D have not indeed the directions of the isothermals a and b, but anything which depends upon an infinitesimal portion of a line is independent of the direction of the line. Therefore the closed line CD may be considered to be a very fine zigzag, composed of alternate portions of isothermals and isentropic lines.

294. Entropy. — Let ϕ be a quantity, associated with a substance, such that

$$\Delta\phi = \frac{\Delta H}{T}, \tag{174}$$

in which $\Delta\phi$ is a change in ϕ, and ΔH is an amount of heat absorbed by the substance and T is its absolute temperature. Let any convenient state of the substance be chosen for which the value *zero* is assigned to the quantity ϕ. This state is called the *zero* state of the substance. Compare Art. 92, on potential energy. The value of ϕ for any other state is then

$$\phi = \Sigma\frac{\Delta H}{T}. \tag{175}$$

The summation is extended throughout any reversible process, bringing the substance from the zero state to the given state. The quantity ϕ is called the *entropy* of the substance. The entropy of a substance in a given state is proportional to the mass of the substance, for the heats, ΔH, absorbed in each step of the process which brings the substance from the zero state to the given state, are proportional to the mass of the substance, so that $\Sigma\frac{\Delta H}{T}$ is also proportional to the mass.

295. Proposition. — *The entropy, ϕ, of a substance has a definite value for each state of the substance.*

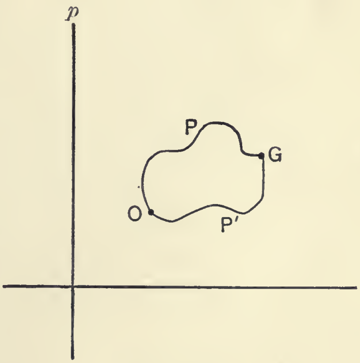

Fig. 141.

Proof. — Let O (Fig. 141) be the zero state, and G the given state of the substance. Let P and P' be any two reversible processes leading from O to G. Let ϕ be equal to the sum $\Sigma\frac{\Delta H}{T}$ for the process P, and let ϕ' be equal to the sum $\Sigma\frac{\Delta H}{T}$ for the process P'. Then $-\phi'$ is the value of $\Sigma\frac{\Delta H}{T}$ for the process P' reversed, and $\phi-\phi'$ is the value of $\Sigma\frac{\Delta H}{T}$ for the cycle $OPGP'$. But from Art. 293, this must be equal to zero; therefore $\phi=\phi'$. Q.E.D.

Remark. — According to the equation $\Delta\phi=\frac{\Delta H}{T}$ (174), the entropy, ϕ, of a substance cannot change, unless the substance gives up heat to, or receives heat from, surrounding bodies. This equation (174), however, takes no account of *sweeping processes*. Indeed, for such processes we have the proposition:

296. Proposition. — *The entropy of a substance is always increased by a sweeping process.*

Proof. — Let the substance be in a state of thermal equilibrium designated by (I), and let its entropy be ϕ_1 before the sweep. After the sweep let it be in a state of thermal equilibrium designated by (II), and let its entropy be ϕ_2. Then

$$\phi_1-\phi_2=\Sigma\frac{\Delta H}{T}, \qquad (a)$$

where the summation is extended throughout a *reversible* process, carrying the substance from II back to I. The sweep and this reversible process together form a cycle.

A sweeping process is essentially independent of external action on a substance * (Art. 276). We may therefore assume that the substance during the sweep neither absorbs nor loses heat, nor does external work, nor has work done upon it. The reversible process, on the other hand, must be accompanied by external action (see Art. 274). Let W be the work done *upon* the substance, and H the heat absorbed by the substance during this reversible process. Then, from the first law of thermodynamics,

$$W=-H. \qquad (b)$$

---

* All the ideas in this chapter are applicable to a *system* of homogeneous substances. The *system* in the case before us is to include all substances which participate in the given sweep.

If W is negative and H positive, then each time the cycle is performed, the heat H is changed into work which is available for outside purposes. This transformation of heat directly into work without compensation is impossible, therefore* W must be positive and H negative. That is, the work W is done on the system, and the heat is given off. H being negative, the sum $\Sigma\frac{\Delta H}{T}$ in (a) is necessarily negative, so that ϕ_2 is greater than ϕ_1.

297. Equivalent sweeps. — Two sweeps for which the increment of entropy is the same, are said to be equivalent. Consider two systems A and B. Let a_1 and a_2 be two states of equilibrium of the system A, such that a_2 may be reached from a_1 by a sweep for which the increment of entropy is ϕ. For example, a_1 may refer to two isolated substances *standing* at different temperatures, and a_2 to the final state reached when the substances are brought into contact. Let b_1 and b_2 be two states of equilibrium of the system B, such that b_2 may be reached from b_1 by a sweep for which the increment of entropy is also ϕ. For example, b_1 may refer to a gas confined in a chamber of a closed vessel b_2 to the gas after it escapes into the whole vessel. Consider a reversible process A, leading from a_1 to a_2 (or a_2 to a_1); and a reversible process B, leading from b_2 to b_1 (or b_1 to b_2). It can be shown by a slight modification of the method of Art. 296, that the external influence necessary to the process A is **equal and opposite** to the external influence necessary to the process B. Therefore the *state* of affairs, after a *sweep* from a_1 to a_2, may be changed by a reversible process to the *state* of affairs which would have existed had the *sweep* from b_1 to b_2 taken place instead of the sweep from a_1 to a_2. Two such *sweeps* are thus equivalent.

---

* The direct transformation of heat into work is shown to be contrary to the second law as follows: Let the heat to be transformed into work be drawn from the heater or the cooler of Carnot's apparatus, and let the work be used to drive Carnot's apparatus backwards, then heat will be transferred from cooler to heater without outstanding compensation.

298. Further statements concerning entropy. — (*a*) If a system suffer an increase of entropy $\Delta\phi$ because of any internal change, that change must be of a sweeping character, bringing the system from an unstable condition to a stable condition. *Such a change when possible is inevitable.*

(*b*) **Thermal and chemical equilibrium.** — The entropy of a system in equilibrium is a maximum; for if any small internal change be possible which further increases ϕ, this change must take place, and the system will not be in a state of equilibrium.*

299. Entropy of a gas. — (*a*) **Change of entropy with pressure at constant volume.** — From equation (148), viz., $pv = RT$, we have $v \cdot dp = R \cdot dT$. Multiplying both members by C_v, then dividing by $pv = RT$ member by member, remembering that $C_v dT = dH$, and that $\frac{dH}{T} = d\phi$, we have

$$d\phi = C_v \frac{dp}{p}. \qquad (a)$$

(*b*) **Change of entropy with volume at constant pressure.** — From $pv = RT$ we have $pdv = RdT$. Multiplying both members by C_p, and reducing as in (*a*), we have

$$d\phi = C_p \frac{dv}{v}. \qquad (b)$$

The total change in ϕ, due to changes in both volume and pressure, is

$$d\phi = C_p \frac{dv}{v} + C_v \frac{dp}{p}. \qquad (c)$$

Integrating from $p = 0$, $v = 0$, $\phi = 0$, C_v and C_p being constant, we have

$$\phi = C_p \log v + C_v \log p. \qquad (176)$$

---

* One of the most remarkable developments of modern physics is the application of thermodynamics to the study of thermal and chemical equilibrium. The student will find the writings of Gibbs, of Planck, and of Duhem of extreme interest in this connection.

Dividing this equation by C_v, and comparing the result with equation (162), we see that the constant in that equation is $\frac{\phi}{C_v}$.

300. The fundamental equation of thermodynamics. — The total internal energy E of a substance can change only by the absorption of heat by or the emission of heat from the substance, or by the doing of work upon or by the substance. Let dE be a change in the internal energy of a substance; let $dH = Td\phi$, see equation (174), be the heat absorbed by the substance, and $-pdv$, see equation (152), be the work done upon the substance. Then

$$dE = T \cdot d\phi - pdv \tag{177}$$

is an equation which formulates both laws of thermodynamics, for it recognizes the equivalence of work and heat on the one hand, and assumes the existence of entropy on the other. All the thermal properties of bodies, in so far as they are deducible from the two laws of thermodynamics and are not matters of separate observation, are deducible* from this equation. This equation of course fails† in case of sweeping processes.

301. The steam engine.‡ — The conditions to be satisfied in order that the efficiency of the steam engine may be a maximum, are that the range of temperature of the steam as it passes through the engine be as great as possible (Arts. 289, 290, 291), and that all the processes undergone by the steam be reversible (see 290). In order that the range of temperature of the steam may be great it is to be at high pressure, and consequently hot, when it enters the cylinder and it is to be expanded, as it does work, until its pressure (and temperature) has been lowered as

---

* See Ency. Britannica (9th edition), article on *Thermodynamics*, by P. G. Tait.

† This failure is not essential, for the only logical thing in connection with a given sweep in any case is the mere recounting of its progress, and even this is impracticable.

‡ The student is referred to special treatises for description of the construction and operation of the steam engine.

much as possible. The economy of the furnace sets a more or less definite limit to the temperature of the steam upon entrance to the cylinder, for if the steam is to be very hot the flue gases of the furnace will be even hotter and will carry off great amounts of heat. The cost of large cylinders and the great radiation therefrom sets a practical limit to the amount of expansion.

The irreversible processes undergone by the steam in an engine are as follows:

(*a*) **Wire drawing.** — If the pipes and passages traversed by the steam from the boiler to the engine are small, the pressure in the cylinder with open ports will be lower than boiler pressure, so that the entering steam passes from a region of high pressure into a region of lower pressure. Also, as the cut-off valve closes, steam will rush into the cylinder through a narrowing aperture. To provide against loss of efficiency from this cause the pipes must be of ample size and the cut-off must be very quick.

(*b*) **Radiation.** — The cooling of pipes and cylinder by radiation — *i.e.* giving up heat to surrounding cooler bodies — is irreversible, and is to be avoided as much as possible by the use of an insulating covering.

(*c*) **Cylinder condensation.** — As one charge of steam in the cylinder expands it cools, and cools the cylinder and piston, so that when steam is next admitted it heats cylinder and piston up again and is itself cooled. This cannot be prevented, but its evil effects can be in large part avoided by having a series of cylinders (a compound engine), so that the range of pressure and the range of temperature in each may not be great; and by providing separate passages for the ingress and egress of steam.

(*d*) **Effect of high piston velocity.** — If piston speed is too great, the pressure of the expanding steam becomes ineffective because the portions of the steam near the moving piston are expanded and cooled before the more remote parts of the steam are affected (compare Art. 276).

The actual effect of these *sweeping* actions upon the efficiency of a steam engine is a thing to be determined by experiment. In fact the study of the steam engine is a thing far removed from theoretical thermodynamics, and the only assistance given to the study of the steam engine by thermodynamics is to enable a keener insight into the properties of steam, and to show in a general way the conditions necessary to high efficiency.

302. The transfer of heat. — The transfer of heat contemplated in the definition of higher and lower temperature, and in the statement of the second law of thermodynamics, takes place in three ways, more or less distinct.

In case one body appears to impart heat **directly** to another at a distance, the transfer is said to be by **Radiation.** This phenomenon will be discussed under **Light.**

In case the transfer of heat is by the distinct movement of a hot substance, as of a hot fluid, the transfer is said to be by **convection.**

In case the transfer is between the contiguous parts of a body which is not at the same temperature throughout, the transfer is said to be by **conduction.** Conduction is, no doubt, due to an increase in the velocities of the molecules in a cooler part of the body, as they rebound after collision with the more rapidly moving molecules in a contiguous warmer part of the body. The following articles refer chiefly to conduction:

Temperature gradient. — Let T be the temperature of a body at a given point, and $T+\Delta T$ its temperature at an adjacent point distant Δx from the first; then $\frac{\Delta T}{\Delta x}$ or $\frac{dT}{dx}$ is called the **temperature gradient** at the given point in the direction of Δx. If the temperature gradient is thus found in three different directions at a point, the **vector sum** of the three will be the **resultant temperature gradient** at the point.

A temperature gradient is always accompanied by a flow of heat in the direction of the gradient from hot to cold.

Fourier's law; thermal conductivity. — Let $\dot{H}$ be the number of thermal units flowing per second across the area a at right angles to the temperature gradient at a point. Then (Fourier's law) $\dot{H}$ is for a given substance proportional to the temperature gradient $\frac{dT}{dx}$ and to a, that is,

$$\dot{H} = Ka\frac{dT}{dx}, \tag{178}$$

in which K is the proportionality constant, which is called the **thermal conductivity** of the given substance.

TABLE XVI.

THERMAL CONDUCTIVITY (K); CALORIES $(\text{cm.}^{-1}) \cdot \text{sec.}^{-1} \cdot$ PER °C.

SUBSTANCES.	K.	SUBSTANCES.	K.
Aluminium .	0.34 at 0°	Cork	0.0007 ——
Antimony . .	0.04 at 0°	Glass	0.0013 to 0018 at 15°
Bismuth . .	0.01 at 0°	Ice	0.0056 ——
Brass . . .	0.15 to 0.20 at 0°	Lava	0.00008 at 0°
Carbon . . .	0.0004 ——	Marble	0.0017 at 0°
Copper . . .	0.90 to 1.10 at 0°	Paraffine	0.00014 at 0°
Iron	0.15 to 0.20 at 0°	Vulcanite (hard rubber)	0.00009 at 0°
Lead	0.07 at 7°		
Magnesium .	0.37 at 50°	Alcohol (C_2H_6O) . .	0.0004 at 5°
Silver . . .	1.09 at 0°	Ether ($C_4H_{10}O$) . . .	0.0004 at 5°
Tin	0.14 at 0°	Mercury	0.0148 at 0°
Zinc	0.30 at 0°	Olive oil	0.0004 at 5°
		Water	0.0012 at 0°

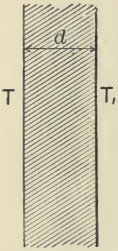

Fig. 142.

303. Example of heat conduction — across a wall. — Consider a wall of thickness d (Fig. 142), of which the thermal conductivity is K; the faces of the wall of area a being maintained at temperatures T and T' respectively. The temperature gradient is $\frac{T-T'}{d}$, so that from equation (178) we have $\dot{H} = Ka\frac{T-T'}{d}$, in which $\dot{H}$ is the number of thermal units per second flowing across the wall.

For discussion of experimental methods for determining thermal conductivity, see Preston, "Theory of Heat," pp. 505–572.

304. Emission of heat. — The transfer of heat across the surface which separates two substances is akin to conduction. The transfer of heat from hot bodies to air and through the air to surrounding bodies is called **emission of heat**.

Newton's law of emissivity. — The rate $\dot{H}$ at which a hot body emits heat is very nearly proportional to the **excess** T of its temperature above that of its surroundings, and roughly proportional to the extent a of its surface. That is,

$$\dot{H} = eaT, \tag{179}$$

in which e is the proportionality factor, which is called the **emissivity** of the surface of the body.

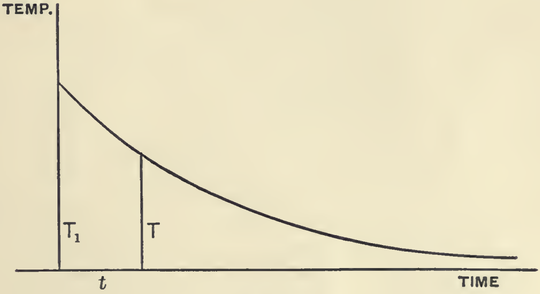

Fig. 143.

305. Cooling of a body by emission. — Consider a body of mass m and of specific heat h. Let a be the superficial area of the body, e its emissivity, and T_1 its temperature excess above its surroundings at an instant ($t=0$), from which time is reckoned, and T its temperature excess after a time t has elapsed. The rate of emission of heat is $\dot{H}=eaT$ from equation (179), so that the heat emitted during a short interval Δt is $eaT \cdot \Delta t$. This heat is extracted from the body as it cools, causing a fall ΔT in its temperature, such that $mh\Delta T = eaT\Delta t$; or since ΔT is negative, we have

$$\frac{dT}{dt} = -\frac{ea}{mh}T.$$

If we integrate, remembering that $T = T_1$ when $t = 0$, we have

$$T = T_1 e^{-\frac{ea}{mh}t}.$$

This is represented graphically in Fig. 143.

The above equation enables the calculation of e when m, h, a, and T_1 are known, and when the temperature T after a known lapse of time t has been observed.

END OF VOLUME I.

INDEX.

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